HIGHWAY BOND CALCULATIONS

UNITED STATES
DEPARTMENT OF AGRICULTURE

ISSUED MAY 1936

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CONTENTS

TYPES OF BONDS

Sinking-fund bonds.—Many highway bonds have been issued as straight terminable bonds to be retired by sinking funds. Some of these bonds were for excessive terms. Although the term varies from 10 to 40 years, most bonds of this type have been within a range of 20 to 30 years. The fund to retire such bonds is accumulated by annual installments paid by the taxpayers and is supposed to draw interest continuously and to accumulate a sufficient amount to discharge the debt at maturity. The interest which the sinking fund draws is usually from 1 to 2 per cent less than the interest paid for the loan. Five per cent highway bonds used to be common with the sinking fund calculated to draw 3½ per cent interest. Table 1 shows the annual payments to the sinking fund necessary to accumulate

[a] This publication consists of selected sections of Department Bulletin 136, Highway Bonds, as published in 1917.

$1,000 at 3, 3½, and 4 per cent compounded semiannually for periods from 1 to 30 years.

TABLE 1.—*Annual payments which, with interest at 3, 3½, and 4 per cent, compounded semiannually, will amount to $1,000 at the end of a term of years.*[1]

Years.	Annual payments.			Years.	Annual payments.		
	3 per cent.	3½ per cent.	4 per cent.		3 per cent.	3½ per cent.	4 per cent.
1	$1,000.0000	$1,000.0000	$1,000.0000	16	$49.5229	$47.5689	$45.6734
2	492.5562	491.3266	490.1000	17	45.8652	43.9283	42.0537
3	323.4583	321.8368	320.2221	18	42.6221	40.7032	38.8504
4	238.9468	237.1428	235.3498	19	39.7280	37.8279	35.9976
5	188.2699	186.3672	184.4796	20	37.1306	35.2499	33.4426
6	154.5102	152.5508	150.6104	21	34.7875	32.9267	31.1429
7	130.4175	128.4252	126.4560	22	32.6639	30.8236	29.0636
8	112.3666	110.3564	108.3733	23	30.7313	28.9116	27.1759
9	98.3436	96.3254	94.3382	24	28.9656	27.1670	25.4557
10	87.1402	85.1208	83.1366	25	27.3469	25.5696	23.8829
11	77.9872	75.9717	73.9954	26	25.8582	24.1024	22.4404
12	70.3721	68.3643	66.3996	27	24.4850	22.7508	21.1136
13	63.9399	61.9427	59.9924	28	23.2149	21.5024	19.8901
14	58.4372	56.4527	54.5191	29	22.0373	20.3465	18.7591
15	53.6780	51.7080	49.7928	30	20.9428	19.2739	17.7113

[1] Example 9, p. 19, shows the method of calculating this table.

Table 2 illustrates how an annual sinking fund of $32,345.83 accumulates for three years to $100,000.

TABLE 2.—*Accumulations of an annual payment of $32,345.83 with interest at 3 per cent compounded semiannually.*

Number of 6-month intervals.	Principal at beginning of 6-month intervals.	Interest during 6-month intervals.	Annual payment at end of 6-month intervals.	Total amount at end of 6-month intervals.
1	$0.00	$0.00	$0.00	$0.00
2	0.00	0.00	32,345.83	32,345.83
3	32,345.83	485.19	0.00	32,831.02
4	32,831.02	492.47	32,345.83	65,669.32
5	65,669.32	985.04	0.00	66,654.36
6	66,654.36	999.81	32,345.83	100,000.00

To obtain the necessary annual payments to produce any multiple of $1 it is necessary merely to multiply the tabular value in Table 1 by the corresponding multiple; thus, an annual sinking fund payment to retire $100,000 in 15 years at 3½ per cent would be $5,170.80. Table 15, pages 42 and 43, gives the yearly or periodic payments necessary to accumulate $1 in a given number of years or periods at varying rates of interest.

There are objections to the sinking-fund method of retiring highway bonds. It may not be possible to obtain continuously the requisite rate of interest on the sinking fund to discharge the debt at maturity. The existence of the sinking fund is a constant temptation to officials to use it for purposes other than the purpose originally intended. If a county, for example, issues bonds for a second object, it is easy to argue that the sinking fund already accumulated may be used to purchase the new securities, and the finances of the com-

munity are in a way to become much confused. This is particularly true since the officers in charge of such operations are frequently changing. Sinking fund tax levies may be deferred through carelessness or under pressure of other needs. The sinking fund always requires careful attention, because it does not progress automatically in most cases.[1] It has sometimes been entirely neglected. The total cost of a bond issue retired by a sinking fund will be greater in the end than the cost of the same bond issue made by either the annuity method or by the serial method.

Annuity bonds.—By the annuity method of issuing bonds both the principal and interest are discharged by constant annual or semi-annual payments. The amount of each payment or installment is determined by the rate of interest and the term of the bond. It usually is necessary to subdivide the bond issue into individual bonds of $100, $500, or $1,000 each. The resulting periodic payment of principal and interest must vary slightly because of this adjustment. Tables 3 and 4 show, in detail, the schedule of principal and interest repayments upon a loan of $100,000 for 20 years, retired by this plan at 4 and 5 per cent per annum, respectively. The necessary adjustment to the nearest $100 bond is also shown. It will be seen that the amount of principal retired is small at first and constantly increases while the interest charge decreases. The sum of interest and principal remains constant, and this is an advantage as the tax is then uniform.

TABLE 3.—*Repayment of a 4 per cent $100,000 loan, including both principal and interest, by a uniform annual payment of $7,358.175 for 20 years.*[2]

Adjusted to nearest cent.				Adjusted to $100 bonds.			
Years.	Principal owing at beginning of year.	Interest for year.	Principal repaid at end of year.	Principal owing at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.
1....	$100,000.00	$4,000.00	$3,358.18	$100,000	$4,000	$3,400	$7,400
2....	96,641.82	3,865.67	3,492.50	96,600	3,864	3,500	7,364
3....	93,149.32	3,725.97	3,632.21	93,100	3,724	3,600	7,324
4....	89,517.11	3,580.68	3,777.49	89,500	3,580	3,800	7,380
5....	85,739.62	3,429.59	3,928.59	85,700	3,428	3,900	7,328
6....	81,811.03	3,272.44	4,085.73	81,800	3,272	4,100	7,372
7....	77,725.30	3,109.01	4,249.17	77,700	3,108	4,200	7,308
8....	73,476.13	2,939.05	4,419.12	73,500	2,940	4,400	7,340
9....	69,057.01	2,762.28	4,595.90	69,100	2,764	4,600	7,364
10....	64,461.11	2,578.44	4,779.73	64,500	2,580	4,800	7,380
11....	59,681.38	2,387.26	4,970.92	59,700	2,388	5,000	7,388
12....	54,710.46	2,188.42	5,169.75	54,700	2,188	5,200	7,388
13....	49,540.71	1,981.63	5,376.55	49,500	1,980	5,400	7,380
14....	44,164.16	1,766.57	5,591.60	44,100	1,764	5,600	7,364
15....	38,572.56	1,542.90	5,815.28	38,500	1,540	5,800	7,340
16....	32,757.28	1,310.29	6,047.88	32,700	1,308	6,000	7,308
17....	26,709.40	1,068.38	6,289.80	26,700	1,068	6,300	7,368
18....	20,419.60	816.78	6,541.39	20,400	816	6,500	7,316
19....	13,878.21	555.13	6,803.05	13,900	556	6,800	7,356
20....	7,075.16	283.01	7,075.16	7,100	284	7,100	7,384
Totals	....	47,163.50	100,000.00	....	47,152	100,000	147,152

[1] In some States there are restrictions on the nature of county investments for sinking fund purposes.
[2] An additional table showing the annual payments necessary to discharge a loan of $1, with interest for varying terms and rates, is given in Table 18 on pp. 48 and 49.

TABLE 4.—*Repayment of a 5 per cent $100,000 loan, including both principal and interest, by a uniform annual payment of $8,024.259* [1] *for 20 years.*

	Adjusted to nearest cent.			Adjusted to $100 bonds.			
Years.	Principal owing at beginning of year.	Interest for year.	Principal repaid at end of year.	Principal owing at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.
1	$100, 000. 00	$5, 000. 00	$3, 024. 25	$100, 000	$5, 000	$3, 000	$8, 000
2	96, 975. 75	4, 848. 79	3, 175. 47	97, 000	4, 850	3, 200	8, 050
3	93, 800. 28	4, 690. 02	3, 334. 24	93, 800	4, 690	3, 300	7, 990
4	90, 466. 04	4, 523. 30	3, 500. 96	90, 500	4, 525	3, 500	8, 025
5	86, 965. 08	4, 348. 25	3, 676. 01	87, 000	4, 350	3, 700	8, 050
6	83, 289. 07	4, 164. 45	3, 859. 81	83, 300	4, 165	3, 900	8, 065
7	79, 429. 26	3, 971. 46	4, 052. 80	79, 400	3, 970	4, 100	8, 070
8	75, 376. 46	3, 768. 82	4, 255. 44	75, 300	3, 765	4, 300	8, 065
9	71, 121. 02	3, 556. 05	4, 468. 21	71, 000	3, 550	4, 500	8, 050
10	66, 652. 81	3, 332. 64	4, 691. 62	66, 500	3, 325	4, 700	8, 025
11	61, 961. 19	3, 098. 06	4, 926. 19	61, 800	3, 090	4, 900	7, 990
12	57, 035. 00	2, 851. 75	5, 172. 51	56, 900	2, 845	5, 200	8, 045
13	51, 862. 49	2, 593. 13	5, 431. 13	51, 700	2, 585	5, 400	7, 985
14	46, 431. 36	2, 321. 57	5, 702. 69	46, 300	2, 315	5, 700	8, 015
15	40, 728. 67	2, 036. 43	5, 987. 83	40, 600	2, 030	6, 000	8, 030
16	34, 740. 84	1, 737. 04	6, 287. 22	34, 600	1, 730	6, 300	8, 030
17	28, 453. 62	1, 422. 68	6, 601. 58	28, 300	1, 415	6, 600	8, 015
18	21, 852. 04	1, 092. 60	6, 931. 66	21, 700	1, 085	6, 900	7, 985
19	14, 920. 38	746. 02	7, 278. 24	14, 800	740	7, 200	7, 940
20	7, 642. 14	382. 12	7, 642. 14	7, 600	380	7, 600	7, 980
Totals		60, 485. 18	100, 000. 00		60, 405	100, 000	160, 405

[1] Cf. Example 14, p. 22, for details of calculations.

Serial bonds.—The serial bond differs somewhat from the annuity bond, because, instead of keeping the annual payment of both principal and interest constant, the principal alone retired each year remains fixed. This type of bond has become more common for highway purposes in recent years.

In Tables 5 and 6 are given the necessary annual payments of interest and principal for an issue of $100,000 for 20 years at 4 and 5 per cent, respectively, where the bonds are retired by annual payments of $5,000 each. The first retirement is sometimes deferred for a number of years.

TABLE 5.—*Schedule of interest and principal to retire a serial loan of $100,000 at 4 per cent, with annual principal repayments of $5,000.*

Years.	Principal outstanding at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.	Years.	Principal outstanding at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.
1	$100, 000	$4, 000	$5, 000	$9, 000	12	$45, 000	$1, 800	$5, 000	$6, 800
2	95, 000	3, 800	5, 000	8, 800	13	40, 000	1, 600	5, 000	6, 600
3	90, 000	3, 600	5, 000	8, 600	14	35, 000	1, 400	5, 000	6, 400
4	85, 000	3, 400	5, 000	8, 400	15	30, 000	1, 200	5, 000	6, 200
5	80, 000	3, 200	5, 000	8, 200	16	25, 000	1, 000	5, 000	6, 000
6	75, 000	3, 000	5, 000	8, 000	17	20, 000	800	5, 000	5, 800
7	70, 000	2, 800	5, 000	7, 800	18	15, 000	600	5, 000	5, 600
8	65, 000	2, 600	5, 000	7, 600	19	10, 000	400	5, 000	5, 400
9	60, 000	2, 400	5, 000	7, 400	20	5, 000	200	5, 000	5, 200
10	55, 000	2, 200	5, 000	7, 200					
11	50, 000	2, 000	5, 000	7, 000	Totals		42, 000	100, 000	142, 000

TABLE 6.—*Schedule of interest and principal to retire a serial loan of $100,000 at 5 per cent, with annual principal repayments of $5,000.*

Years.	Principal outstanding at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.	Years.	Principal outstanding at beginning of year.	Interest for year.	Principal repaid at end of year.	Total.
1	$100,000	$5,000	$5,000	$10,000	12	$45,000	$2,250	$5,000	$7,250
2	95,000	4,750	5,000	9,750	13	40,000	2,000	5,000	7,000
3	90,000	4,500	5,000	9,500	14	35,000	1,750	5,000	6,750
4	85,000	4,250	5,000	9,250	15	30,000	1,500	5,000	6,500
5	80,000	4,000	5,000	9,000	16	25,000	1,250	5,000	6,250
6	75,000	3,750	5,000	8,750	17	20,000	1,000	5,000	6,000
7	70,000	3,500	5,000	8,500	18	15,000	750	5,000	5,750
8	65,000	3,250	5,000	8,250	19	10,000	500	5,000	5,500
9	60,000	3,000	5,000	8,000	20	5,000	250	5,000	5,250
10	55,000	2,750	5,000	7,750					
11	50,000	2,500	5,000	7,500	Totals.	....	52,500	100,000	152,500

Comparison of serial, annuity, and sinking-fund bonds.—It will be noticed that the total expense to the community under the serial plan is somewhat less than under the annuity plan. The expense by either method is, however, considerably less than the expense under the sinking-fund plan. For the purpose of comparison the total expense to the community under each plan is assembled under Table 7.

Tables 3 to 6, inclusive, are computed with interest payable annually. Bonds with interest payable semiannually sell better. Similar tables or schedules for the annuity and serial plans of bond issues to conform to semiannual interest payments can be easily prepared. Schedules can also be prepared to show the progress of a bond loan when the bonds are bought at a premium or discount.[1]

TABLE 7.—*Total cost of a loan of $100,000 for 20 years, interest compounded annually.*

Annual interest on bonds.	Sinking fund compounded annually at—			Annuity.	Serial.
	3 per cent.	3½ per cent.	4 per cent.		
4	$154,431	$150,722	$147,163	$147,163	$142,000
4½	164,431	160,722	157,163	153,752	147,250
5	174,431	170,722	167,163	160,485	152,500
5½	184,431	180,722	177,163	167,359	157,750
6	194,431	190,722	187,163	174,369	163,000

In a bond issue by any given plan the amount, the interest, and the term may be fixed at will, but when this is done the annual repayments of principal and interest are theoretically determined. Thus, by the annuity method, if $100,000 is to be issued at 5 per cent annually and retired in 20 years, the annual amount of interest and principal is at once determined to be approximately $8,000.

[1] Cf. pages 13 to 37, for details of such schedules.

For the same bond issue under the serial plan, the total annual payment varies because the interest varies, but each yearly payment of interest and principal is nevertheless fixed.

Under the sinking-fund plan the annual payment necessary for principal and interest is theoretically constant, but it depends upon the interest realized upon the sinking fund. It is not safe, as a rule, to estimate this interest at more than 3½ per cent. Then for a $100,000 20-year loan, with annual interest on the sinking fund, the total annual payment would be $8,536.11. If the sinking fund could earn the rate of interest which is paid upon the loan there would be no advantage in expense to the community in the annuity or the serial bond over the sinking-fund bond. There is given in Table 8 the total mill tax on $1 to retire a bond issue of $100,000 by the sinking fund or the annuity plan.

TABLE 8.—*Annual mill tax on $1 for interest and retirement on a bond issue of $100,000, at 5 per cent annual interest, for terms of 10 and 20 years.*

Valuation.	Mill tax.							
	10 years.				20 years.			
	Sinking-fund plan.[1]			Annuity plan.[2]	Sinking-fund plan.[1]			Annuity plan.[2]
	3 per cent.	3½ per cent.	4 per cent.		3 per cent.	3½ per cent.	4 per cent.	
$1,000,000	13.723	13.524	13.329	12.950	8.722	8.536	8.358	8.024
1,500,000	9.149	9.016	8.886	8.634	5.814	5.691	5.572	5.350
2,000,000	6.861	6.762	6.665	6.475	4.361	4.268	4.179	4.012
2,500,000	5.489	5.410	5.332	5.180	3.489	3.414	3.343	3.210
3,000,000	4.574	4.508	4.443	4.317	2.907	2.845	2.786	2.675
3,500,000	3.921	3.864	3.808	3.700	2.492	2.439	2.388	2.293
4,000,000	3.431	3.381	3.332	3.238	2.180	2.134	2.090	2.006
4,500,000	3.050	3.005	2.962	2.878	1.938	1.897	1.857	1.783
5,000,000	2.745	2.705	2.666	2.590	1.744	1.707	1.672	1.605
5,500,000	2.495	2.459	2.423	2.355	1.586	1.552	1.520	1.459
6,000,000	2.287	2.254	2.222	2.158	1.454	1.423	1.393	1.337
6,500,000	2.111	2.081	2.051	1.992	1.342	1.313	1.286	1.235
7,000,000	1.960	1.932	1.904	1.850	1.246	1.219	1.194	1.146
7,500,000	1.830	1.803	1.777	1.727	1.163	1.138	1.114	1.070
8,000,000	1.715	1.691	1.666	1.619	1.090	1.067	1.045	1.003
8,500,000	1.614	1.591	1.568	1.524	1.026	1.004	.983	.944
9,000,000	1.525	1.503	1.481	1.439	.969	.948	.929	.892
9,500,000	1.445	1.424	1.403	1.363	.918	.899	.880	.845
10,000,000	1.372	1.352	1.333	1.295	.872	.854	.836	.802

[1] With interest compounded annually.
[2] The tax for the serial plan is slightly less, but varies from year to year.

It is quite probable that so many 30-year bonds are issued in order to take advantage of the fact that bonds of that term result in a low annual charge for interest and sinking fund. It will be seen from Table 9 that very little advantage is gained by fixing the term of a bond longer than 30 years. The annual charge decreases very slowly from that point, whereas the total charge increases rapidly.

TABLE 9.—*Annual and total costs of a loan of $100,000 for varying periods, with sinking fund to draw 3½ per cent interest, compounded annually.*

Term in years.	Annual interest on bonds. 4 per cent. Total annual payment, interest, and sinking fund.	4 per cent. Total cost of loan.	5 per cent. Total annual payment, interest, and sinking fund.	5 per cent. Total cost of loan.
5	$22,648	$113,241	$23,648	$118,241
10	12,524	125,241	13,524	135,241
15	9,183	137,738	10,183	152,738
20	7,536	150,722	8,536	170,722
25	6,567	164,185	7,567	189,185
30	5,937	178,114	6,937	208,114
35	5,500	192,494	6,500	227,494
40	5,183	207,309	6,183	247,309
45	4,945	222,540	5,945	267,540
50	4,763	238,169	5,763	288,169

The curves of annual cost of interest and retirement fall very slowly after the 30-year point.

It is an unfortunate fact that some highways do not have a life of 30 years, and it is quite evident that the life of the highway and not the apparent economic term of the bond should determine the length of the loan.

There is a further advantage in the annuity or serial bond for highway construction, because it is more likely under such a bond that the road surface will be paid for before it is entirely worn out. If an annuity or serial bond begins to mature immediately, this is not considered a serious objection among bankers. These types of bonds are particularly adapted for financing operations which by their very nature involve a wasting of the property. A highway is in part a wasting property and it is desirable to have established a margin of safety in highway financing.

From the nature of the annuity or the serial form of highway bonds it is never necessary to issue new or refunding bonds at the end of the term. Both of these types of bonds have the advantage that they accomplish with one financial operation all that the sinking-fund type of bond can accomplish. The main advantage, however, of both types of bonds is that the community saves more money than under the sinking-fund plan because it avoids paying a higher rate on borrowed money than it can obtain on money that it loans.

Highway bonds are seldom sold at par. Not infrequently they command a slight premium; that is to say, they are sold at an advance over the par value. In nearly every State the law provides that municipal bonds shall not be sold at less than par. When the purchaser pays a premium for a 5 per cent highway bond it will yield

less than 5 per cent. To enable investors to determine quickly the net rate of yield from a bond purchased at a premium or at a discount, tables known as bond tables have been calculated. On pages 50 and 51 is presented a short bond table of this kind (Table 19). From this table the net yield of a bond with a nominal rate of interest of from 3 to 6 per cent, payable semiannually and for varying terms, may be calculated for various prices. Thus a 5 per cent 15-year highway bond purchased at 103.20, or with a premium of 3.20 per cent, will be found to yield the purchaser 4.70 per cent on his investment. Such tables are of more important interest to the purchaser than to the municipality offering the bonds, but they are necessary for the intelligent direction of the bond issue.

In calculating the price to be paid for serial bonds, it is customary to treat each series separately and to find the price that yields the given net rate by adding the separate prices. Some formulas will be found, however, in the treatment of highway bond calculations which considerably shorten the labor of calculating the price to be paid for serial bonds and the labor of related calculations.

Special form of annuity bond.—In the operation of the annuity bond both interest and principal are discharged by a series of equal installments, usually semiannual. Each installment contains interest on the bonds outstanding at the beginning of the interval and the balance is applied to retire the bonds. The effect of this method is to diminish steadily the investment of the purchaser. If, however, the borrower should arrange to set aside periodically in a sinking fund a fixed sum *in excess of the periodic interest on the entire issue,* the effect would be to leave the total investment of the purchaser undisturbed until the sinking fund had accumulated to the amount of the loan. When the excess of the periodic installment over the required interest is arbitrarily selected and accumulates at a given rate of interest, the term of the bond is thereby absolutely fixed. A simple way to accomplish this result is to add to the nominal interest rate which the bonds pay a percentage of the principal to be set aside in a sinking fund to retire the bonds. There is produced thus a new nominal rate. Since both interest and principal are discharged by the periodical payment of interest or dividends at the new nominal rate, an issue of this character may be described as a special form of annuity bond.

Table 10 shows the resulting terms in years of a bond issue for \$1,000,000 where from 1½ to one-half per cent of the principal is set aside semiannually in a sinking fund which draws 3 per cent compounded semiannually. The original interest rate on the bonds is assumed to be 3 per cent, payable semiannually, and the new increased nominal rate varies then from 6 to 4 per cent. The last col-

umn shows the total cost to the borrower for the loan of $1,000,000 under this method.

TABLE 10.—*Necessary terms and total costs of a bond issue of $1,000,000 at 3 per cent, payable semiannually, when retired by various arbitrary fractions of the principal set aside and compounded semiannually.*

Applied semiannually to sinking fund to retire bond issue.	New increased interest rate on original 3% bonds.	Term of bonds.	Total cost to borrower.
Per cent. of loan.	*Per cent.*	*Years.*	*Dollars.*
1½	6	23½	1,410,000
1⅜	5¾	25	1,437,500
1¼	5½	26½	1,457,500
1⅛	5¼	28½	1,496,250
1	5	31	1,550,000
⅞	4¾	34	1,615,000
¾	4½	37	1,665,000
⅝	4¼	41½	1,763,750
½	4	47	1,880,000
½	4	50	2,000,000

The progress of the accumulation of the semiannual sinking fund under the plan here outlined is shown for varying retirement rates in Table 12. It is possible so to determine the rate of retirement that the resulting term of the bonds is integral instead of fractional. The increased nominal rates for 3 per cent bonds to retire in varying integral terms are as follows: [1]

TABLE 11.—*Equivalent nominal rates for retiring 3 per cent bonds in varying terms.*

	Per cent.		Per cent.
10 years	11.649148	30 years	5.078686
20 years	6.685420	40 years	4.309664
25 years	5.714336	50 years	3.874114

The details of advertising and selling highway bonds are frequently prescribed by law. Bids from bond houses are always made conditioned on an investigation of the validity of all proceedings leading to the issue. The attorneys for the bidders will require from the municipality certified copies of all papers concerning the transaction. There frequently is much variation in the form of the bids for a single issue. The items of denomination of the bonds, options on delivery, portion of the issue bid for, deposit of the money in stipulated banks, and items of less importance are often written into the bids.

[1] This rate per cent is determined by the formula:

$$\text{Rate per cent} = 3 + 200/s_{\overline{2n}|}$$

where n is the number of years and $s_{\overline{2n}|}$ is determined from Table 14, page 40, at the rate of 1½%.

TABLE 12.—*Accumulations at 3 per cent, convertible semiannually, of a semiannual sinking fund to extinguish a loan of $1,000,000.*

New increased nominal rate on original 3% bonds.	4 per cent.	4¼ per cent.	4½ per cent.	4¾ per cent.	5 per cent.	5¼ per cent.	5½ per cent.	5¾ per cent.	6 per cent.
	Percentage which the semiannual sinking-fund payment bears to the loan.								
Years.	½ per cent.	⅝ per cent.	¾ per cent.	⅞ per cent.	1 per cent.	1⅛ per cent.	1¼ per cent.	1⅜ per cent.	1½ per cent.
0.5	$5,000.00	$6,250.00	$7,500.00	$8,750.00	$10,000.00	$11,250.00	$12,500.00	$13,750.00	$15,000.00
1.0	10,075.00	12,593.75	15,112.50	17,631.25	20,150.00	22,668.75	25,187.50	27,706.25	30,225.00
1.5	15,226.13	19,032.66	22,839.19	26,645.72	30,452.25	34,258.78	38,065.31	41,871.84	45,678.38
2.0	20,454.52	25,568.15	30,681.78	35,795.40	40,909.03	46,022.66	51,136.29	56,249.92	61,363.55
5.0	53,513.61	66,892.01	80,270.41	93,648.81	107,027.22	120,405.62	133,784.02	147,162.42	160,540.83
10.0	115,618.34	144,522.92	173,427.50	202,332.09	231,236.67	260,141.25	289,045.84	317,950.42	346,855.01
15.0	187,693.41	234,616.76	281,540.11	328,463.46	375,386.81	422,310.17	469,233.52	516,156.87	563,080.22
20.0	271,339.47	339,174.34	407,009.20	474,844.07	542,678.94	610,513.81	678,348.67	746,183.54	814,018.41
23.0	327,842.07	409,802.59	491,763.10	573,723.62	655,684.14	737,644.66	819,605.17	901,565.69	983,526.21
23.5	337,759.70	422,199.63	506,639.55	591,079.48	675,519.40	759,959.33	844,399.25	928,839.18	1,013,279.10
24.5	358,043.49	447,554.36	537,065.23	626,576.10	716,086.98	805,597.85	895,108.72	984,619.59	
25.0	368,414.14	460,517.68	552,621.21	644,724.75	736,828.28	828,931.82	921,035.35	1,013,138.89	
26.0	389,624.46	487,030.57	584,436.69	681,842.80	779,248.92	876,655.03	974,061.14		
26.5	400,468.82	500,586.03	600,703.24	700,820.44	800,937.65	901,054.86	1,001,172.06		
28.0	433,987.71	542,484.64	650,981.57	759,478.50	867,975.43	976,472.36			
28.5	445,497.53	556,871.91	668,246.30	779,620.68	890,995.06	1,002,369.44			
30.0	481,073.26	601,341.57	721,609.89	841,878.20	962,146.52				
30.5	493,289.36	616,611.70	739,934.04	863,256.38	986,578.71				
31.0	505,688.70	632,110.87	758,533.05	884,955.22	1,011,377.40				
33.5	570,531.68	713,164.60	855,797.52	998,430.44					
34.0	584,089.65	730,112.07	876,134.48	1,022,156.90					
35.0	611,818.76	764,773.46	917,728.15						
36.5	654,991.78	818,739.72	982,487.67						
37.0	669,816.65	837,270.82	1,004,724.98						
40.0	763,554.26	954,442.83							
41.0	796,707.69	995,884.61							
41.5	813,658.31	1,017,072.88							
45.0	939,649.50								
46.5	997,797.29								
47.0	1,017,764.25								
50.0	1,144,015.22								

BOND CALCULATIONS

Introduction.—This section presents briefly the application of the theory of compound interest to highway bonds. There are six important quantities in terms of which the solution of most problems can be expressed. If i is the rate of interest and n the term of years, these quantities are:

The accumulation of 1 at the end of n years, r^n;

The accumulation of an annuity of 1 per annum at the end of n years, $s_{\overline{n}|}$;

The annual sinking fund which will accumulate to 1 at the end of n years, $1/s_{\overline{n}|}$;

The present value of 1 due in n years, v^n;

The present value of an annuity of 1 per annum for n years, $a_{\overline{n}|}$; and

The annuity for n years which 1 will purchase, or the annuity necessary to discharge a debt of 1 in n years with interest, $1/a_{\overline{n}|}$.

The first three are accumulative functions and the last three are discount or present value functions.

The mathematical formulas for these six quantities are:

$$r^n = (1+i)^n \qquad s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \qquad \frac{1}{s_{\overline{n}|}} = \frac{i}{(1+i)^n - 1}$$

$$v^n = (1+i)^{-n} \qquad a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{i} \qquad \frac{1}{a_{\overline{n}|}} = \frac{i}{1-(1+i)^{-n}}$$

The values of most of these functions are given more or less completely in published tables of interest.[1]

Definitions.—*Interest* may be defined as the consideration for the use of capital. The capital is called the *principal.*

The *rate* at which a given principal is earning interest requires the adoption of some interval as the unit of time, and this is usually the *year.*

It is clear that interest when received may be added to the principal and so in turn earn interest. This process, called *compounding,* takes place at the end of *stated intervals,* as every three months, six months, or year.

[1] Short tables to seven places for 60 intervals and 14 interest rates are given on pages 38 to 51.

Mathematical rates.—The *effective rate of interest* is the interest earned by one unit of principal (one dollar) in one unit of time (one year) when interest *is compounded* at the end of each *stated interval.*

The *nominal rate of interest* is the total interest earned by one unit of principal (one dollar) in one unit of time (one year) when interest *is not compounded* at the end of each *stated interval.*

It follows that the nominal and effective rates of interest coincide only when the *stated interval* is the unit of time (one year).

Commercial rate.—In commercial transactions the rate of interest is usually quoted as a rate *per cent,* or per hundred units of principal, instead of a rate *per unit* of principal, as in the above definitions. To find the mathematical rate as above defined, divide the commercial rate by 100. For example, the mathematical rate corresponding to the commercial rate 6 per cent is 6/100, or .06. The mathematical rate is used in the following formulas.

Relation between effective and nominal rates of interest.—In any transaction there is an effective rate of interest i and a corresponding nominal rate of interest j. This relation can be expressed by an algebraic formula which involves the number of *stated intervals, m,* in one year. At the nominal rate j, during each stated interval $1/m$th of a year in length, one unit of principal would earn j/m in interest which, added to the unit, gives an amount $1+j/m$. If the principal 1 accumulates in the first interval to $1+j/m$, it follows by proportion that the principal $1+j/m$ would accumulate in the second interval to $(1+j/m)^2$. In like manner, at the end of the mth interval, the accumulation would be $(1+j/m)^m$. The total interest earned in the m intervals, or one year, is the difference between the accumulation and the original unit of principal, which by definition is the *effective rate* of interest i. Hence the fundamental formula:

$$\boldsymbol{i}=(1+\boldsymbol{j}/\boldsymbol{m})^{\boldsymbol{m}}-1 \qquad (1)$$

or

$$1+\boldsymbol{i}=(1+\boldsymbol{j}/\boldsymbol{m})^{\boldsymbol{m}}. \qquad (2)$$

Solving for j, there results

$$\boldsymbol{j}=\boldsymbol{m}[(1+\boldsymbol{i})^{1/\boldsymbol{m}}-1]. \qquad (3)$$

The number of times, m, that interest is added, or converted into principal each year, is the *frequency of conversion.* A nominal rate of interest, convertible m times a year, is indicated by the symbol $j_{(m)}$.

Example 1.—The nominal rate of interest j on deposits is 3% and interest is added to the principal every six months; to find the effective rate i.

Here $j=.03$ and $m=2$. From formula (1) there results

$$i=(1+.03/2)^2-1=(1.015)^2-1=.030225.$$

The effective rate 3.0225% is thus slightly higher than the corresponding nominal rate convertible twice per annum.

Example 2.—The effective rate of interest is 6%; to find the corresponding nominal rate when interest is convertible semiannually.

Here i and m are given to find j; hence from formula (3) there results

$$j=2[(1+.06)^{\frac{1}{2}}-1]=2(1.06)^{\frac{1}{2}}-2=2.059126-2=.059126.$$

It is necessary to extract the square root of 1.06. The final result shows that $j=5.9126\%$, and again the nominal rate is smaller than the corresponding effective rate.

Amount of* 1 *in n years at compound interest.—Let the effective rate of interest be i. At the end of the first year the accumulation is $1+i$. During the second year this principal $1+i$ will be increased in the ratio of 1 to $1+i$, and will therefore amount at the end of the second year to $(1+i)(1+i)$, or $(1+i)^2$. In this way at the end of n years the amount is $(1+i)^n$.

Let P be the principal and S the amount of P at the end of n years at compound interest at the effective rate i. Since 1 amounts to $(1+i)^n$ in n years, P would amount to $P(1+i)^n$. There results, therefore, the formula

$$\boldsymbol{S=P(1+i)^n}. \tag{4}$$

Hence

$$\boldsymbol{P=S/(1+i)^n=Sv^n}, \tag{5}$$

where

$$\boldsymbol{v=1/(1+i)}. \tag{6}$$

If in the above formulas $1+i$ is replaced by $(1+j/m)^m$, to which it is equivalent according to formula (2), it follows that

$$\boldsymbol{S=P(1+j/m)^{mn}}, \tag{7}$$

and

$$\boldsymbol{P=S/(1+j/m)^{mn}=Sv^{mn}}, \tag{8}$$

where

$$\boldsymbol{v=1/(1+j/m)}. \tag{9}$$

These formulas express the relation between P and S in terms of the nominal rate j and the frequency of conversion m. The values to seven places of decimals of $(1+i)^n$ and v^n for various rates of interest and for 60 intervals or years are given in Tables 13 and 16.

Example 3.—To find the amount of \$12,375 at 3% compound interest in 30 years.

By formula (4)

$$S=(1+.03)^{30}\times\$12,375=2.4272625\times\$12,375=\$30,037.37.$$

The value of $(1.03)^{30}$ was taken from Table 13.

Example 4.—\$12,375 is placed in a bank; to find the amount in 30 years if interest is 3% and is compounded semiannually.

The nominal rate of 3%, convertible twice a year, requires formula (7) with $j=.03$ and $m=2$. Substituting, the result is:

$$S=(1+.03/2)^{2\times30}\times\$12,375=(1.015)^{60}\times\$12,375=2.4432198\times\$12,375=\$30,234.85.$$

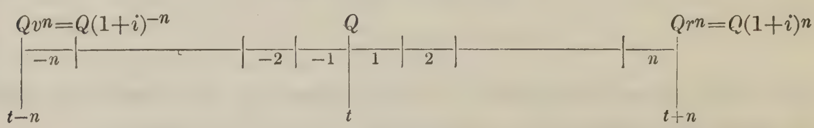
The discount factor.—Because of the power of money to earn interest the value of money depends upon the time to which it is referred. Then in order to compare sums of money due at different times, they must be referred to the *same point in time*.

Formula (5) gives the principal P which will accumulate at the effective rate i in n years to the amount S. If $S=1$ and $n=1$, the formula gives the present value of 1 due in one year. This is usually denoted by the symbol v, so that

$$v=1/(1+i)=(1+i)^{-1}.$$

Similarly $v^2=1/(1+i)^2=(1+i)^{-2}$ is the present value of 1 due in two years, and $v^n=1/(1+i)^n=(1+i)^{-n}$ is the present value of 1 due in n years.

The symbol v is often called the discount factor, and if it is desired to find the value of money *n years before the point in time under consideration*, it is necessary only to multiply the quantity by v^n. The factor $1+i$, which is frequently denoted by r, is accumulative in character, and formula (4) shows that, to find the value of a quantity of money Q, *n years after the point in time under consideration*, it is necessary merely to multiply the quantity by $(1+i)^n$.



More generally, *when i is the effective rate per interval*, the value of Q, at a time n intervals after the point t, is $Q(1+i)^n=Qr^n$, and its value n intervals before point t is $Q(1+i)^{-n}=Qv^n$.

Annuities-certain.—An *annuity* is a series of payments made at *equal intervals* during the continuance of a given *status*.

The *status*, or condition of payment of the annuity, may take a variety of forms. If the status is a fixed term of years, the annuity is an *annuity-certain*. Thus payments of one hundred dollars a year for ten years constitute an annuity-certain. The sum of the payments on an annuity in one year, when the payments are of the same amount, is the *annual rent*.

Payments of twenty-five dollars are made at the end of every month for ten years. This is an annuity-certain with an annual rent of three hundred dollars.

When payments are made at the *end* of each interval, the annuity-certain is said to be *immediate*; when payments are made at the *beginning* of each interval, the annuity-certain is said to be *due*.

Amount of an immediate annuity-certain.—The value of an annuity at the end of its term is called the *amount*. The amount of an immediate annuity-certain for n years with an annual rent 1,

payable at the end of each year, is designated by the symbol $s_{\overline{n}|}$. To find $s_{\overline{n}|}$ each annual payment must be accumulated, at the effective rate of interest i, to the end of the term of the annuity. The first payment of 1 accumulates in $n-1$ years to $(1+i)^{n-1}$; the second payment of 1, in $n-2$ years, to $(1+i)^{n-2}$; etc. . . . . ; the $(n-1)$th payment of 1, in 1 year, to $(1+i)$; and the nth payment at the end of the term is 1. Adding the separate amounts in reverse order, there results

$$s_{\overline{n}|}=1+(1+i)+(1+i)^2+\ldots\ldots\ldots\ldots+(1+i)^{n-1}.$$

The sum of this geometric series is

$$s_{\overline{n}|}=\frac{(1+i)^n-1}{i}. \tag{10}$$

Values of this quantity are given for various rates of interest and terms in Table 14.

Example 5.—To find the accumulation in 47 years of an annual sinking fund of 1% of \$1,000,000, if the fund is credited annually with 3% compound interest.

This is an application of formula (10) where $n=47$ and $i=.03$; since $s_{\overline{47}|}=100.3965009$ the accumulation will be

$$100.3965009\times\$10{,}000=\$1{,}003{,}965.01.$$

The same principles may be applied to find the amount of an annuity for n years with annual rent 1 payable in p equal installments during each year. The amount of such an annuity is designated by the symbol $s_{\overline{n}|}^{(p)}$, and its value is represented by the following formula:

$$s_{\overline{n}|}^{(p)}=\frac{(1+i)^n-1}{p[(1+i)^{1/p}-1]}. \tag{11}$$

If $1+i$ is replaced by $(1+j/m)^m$ in accordance with formula (2), the amount of the annuity is then expressed in terms of the nominal rate of interest j with frequency of conversion m, thus:

$$s_{\overline{n}|}=\frac{(1+j/m)^{mn}-1}{(1+j/m)^m-1}, \tag{12}$$

$$s_{\overline{n}|}^{(p)}=\frac{(1+j/m)^{mn}-1}{p[(1+j/m)^{m/p}-1]}. \tag{13}$$

Example 6.—What will be the accumulation in 47 years of an annual sinking fund of 1% of \$1,000,000, paid in semiannually, if the fund is credited as received with 3% interest compounded annually?

This is an application of formula (11) where $n=47$, $p=2$, $i=.03$, hence

$$s_{\overline{47}|}^{(2)}=\frac{(1+.03)^{47}-1}{2\,[(1+.03)^{\frac{1}{2}}-1]}=\frac{3.0118950}{.0297783}=101.143954$$

and the accumulation will be $101.143954\times\$10{,}000=\$1{,}011{,}439.54$.

The special case where interest is converted with the same frequency as the payment of annuity installments, or when $m=p$, deserves particular mention. Formula (13) then reduces to

$$s_{\overline{n}|}^{(p)}=\frac{1}{p}\cdot\frac{(1+j/p)^{np}-1}{j/p}=\frac{1}{p}\cdot s_{\overline{np}|}, \tag{14}$$

where $s_{\overline{np}|}$ is to be taken at the effective rate j/p.

Example 7.—What will be the accumulation in 47 years of an annual sinking fund of 1% of \$1,000,000, paid in semiannually, if the fund is credited with a nominal rate of 3% convertible twice a year?

This is an application of formula (14) where $n=47$, $p=2$, and $s_{\overline{94}|}$ is taken at $1\frac{1}{2}\%$; hence

$$\tfrac{1}{2}s_{\overline{94}|}\times \$10{,}000=\$1{,}017{,}764.25.$$[1]

Sinking fund which will amount to 1.—An annuity with annual rent of 1 will amount in n years to $s_{\overline{n}|}$; it follows that an annuity with annual rent of $1/s_{\overline{n}|}$ will amount in n years to 1. The quantity $1/s_{\overline{n}|}$ is the *sinking fund* which will accumulate to 1 in n years.

Values for this important function

$$\frac{1}{s_{\overline{n}|}}=\frac{i}{(1+i)^n-1} \tag{15}$$

are given for various rates of interest and for terms ranging from 1 to 60 intervals or years in Table 15.

Example 8.—To find an annual sinking fund, which, credited with 3% compound interest, will accumulate in 50 years to \$1,000,000.

Applying formula (15) where $n=50$, and $i=.03$, there results $1/s_{\overline{50}|}=.0088655$. Therefore the required sinking fund is

$$.0088655\times \$1{,}000{,}000=\$8{,}865.50.$$

In like manner $1/s_{\overline{n}|}^{(p)}$ is the annual rent of an annuity which, at the nominal rate j convertible m times a year, will accumulate to 1 in n years. The annual rent is payable in p installments during each year; hence each installment is equal to $1/ps_{\overline{n}|}^{(p)}$. The installments may be regarded as the sinking fund, payable at the end of every pth part of a year, which in n years will accumulate to 1. The amount of each payment to the sinking fund is

$$\frac{1}{ps_{\overline{n}|}^{(p)}}=\frac{(1+j/m)^{m/p}-1}{(1+j/m)^{mn}-1}. \tag{16}$$

When $p=1$ and $m=2$, formula (16) gives the value of the *annual* sinking fund which, improved at compound interest *semiannually*, will accumulate in n years to 1.

The formula simplifies to the following:

$$\frac{(1+j/2)^2-1}{j/2}\cdot\frac{j/2}{(1+j/2)^{2n}-1}=s_{\overline{2}|}\cdot\frac{1}{s_{\overline{2n}|}}\quad\text{at rate } j/2. \tag{17}$$

This formula is of considerable practical importance because payments to the sinking fund are usually made annually and the fund

[1] For calculation of $s_{\overline{94}|}$ see Example 22, page 34.

credited with interest semiannually. Table 1, on page 4, was calculated by formula (17).

Example 9.—To find the annual payment which will accumulate in 20 years to \$100,000 when interest is $3\frac{1}{2}$ per cent compounded semiannually.

Taking $n=20$ and $j=.035$ and consulting Tables 14 and 15 with $1\frac{3}{4}$ per cent interest for values of $s_{\overline{2}|}$ and $1/s_{\overline{40}|}$, respectively, there results:

$$s_{\overline{2}|} \cdot \frac{1}{s_{\overline{40}|}} = 2.0175 \times .0174721 = .0352500.$$

Hence the annual payment to sinking fund is

$$.0352500 \times \$100{,}000 = \$3{,}525.00.$$

Example 10.—To find the sinking fund, which set aside semiannually and accumulated as received, with 3% compound interest, will amount in 50 years to \$1,000,000.

Here formula (16) is used with $p=2$, $m=1$, $j=.03$, $n=50$, and

$$\frac{1}{2s^{(2)}_{\overline{50}|}} = \frac{(1+.03)^{\frac{1}{2}}-1}{(1+.03)^{50}-1} = .00439999.$$

The required sinking fund is therefore

$$.00439999 \times \$1{,}000{,}000 = \$4{,}399.99.$$

In the special case when the frequency of conversion coincides with the number of payments per annum, or $m=p$, the amount of *each payment* to the sinking fund is

$$\frac{1}{ps^{(p)}_{\overline{n}|}} = \frac{j/p}{(1+j/p)^{np}-1} = \frac{1}{s_{\overline{np}|}}, \tag{18}$$

where $s_{\overline{np}|}$ is to be taken at rate j/p.

Example 11.—To find a sinking fund which, set aside semiannually and credited with a nominal rate of 3% convertible twice a year, will accumulate in 30 years to \$1,000,000.

Here apply formula (18) by substituting $p=2$, $j=.03$, and $n=30$; this gives

$$\frac{1}{2s^{(2)}_{\overline{30}|}} = \frac{1}{s_{\overline{60}|}} = .0103934,$$

where $1/s_{\overline{60}|}$ is taken at $1\frac{1}{2}$%. Then the sinking fund which would accumulate to \$1,000,000 is

$$.0103934 \times \$1{,}000{,}000 = \$10{,}393.40.$$

Four important cases of sinking funds are illustrated in the preceding examples. They arise from the fact that payments to a sinking fund may be annual or semiannual and interest on a sinking fund annual or semiannual. Formula (16) covers all of them when p and m are properly chosen. The following schedule illustrates this fact:

Case.	p	Sinking-fund payments.	m	Interest on sinking fund.	Illustrated in example.
1	1	Annual	1	Annual	8
2	1	Annual	2	Semiannual	9
3	2	Semiannual	1	Annual	10
4	2	Semiannual	2	Semiannual	11

In most cases in the illustrative tables in the body of this bulletin, for simplicity of presentation, annual payments and annual interest are assumed, whereas in practice usually annual payments and semi-annual interest are employed.

Present value of an immediate annuity-certain.—The present value of an immediate annuity-certain for n years, with annual rent 1 payable at the end of each year, is designated by the symbol $a_{\overline{n}|}$. It is equal to the sum of the present values of 1, due at the succeeding yearly intervals. By formula (5) the present value of 1, due at the end of one year at the effective rate of interest i, is $v=1/(1+i)$; at the end of two years, $v^2=1/(1+i)^2$, etc. . . . .; at the end of n years, $v^n=1/(1+i)^n$. Hence

$$a_{\overline{n}|}=v+v^2+\ldots\ldots\ldots+v^n$$

$$=\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots\ldots\ldots+\frac{1}{(1+i)^n}.$$

The sum of this geometric series is

$$a_{\overline{n}|}=\frac{1-v^n}{i}=\frac{1-(1+i)^{-n}}{i} \tag{19}$$

and its values are given in Table 17.

Example 12.—To find the present value at 3% of an annual payment of $56,325 at the end of each year for thirty years.

Referring to Table 17, it is seen that $a_{\overline{30}|}$ at 3% is 19.6004413, and therefore the equired present value is

$$19.6004413\times\$56,325=\$1,103,994.86.$$

While the above demonstration relates to an annuity of 1 per annum, payable at the end of each year, the same principles apply to finding the present value of an annuity of 1 per annum, payable in p installments during each year. The present value of such an annuity is designated by the symbol $a^{(p)}_{\overline{n}|}$, and its value is represented by the following formula:

$$a^{(p)}_{\overline{n}|}=\frac{1-v^n}{p[(1+i)^{1/p}-1]}=\frac{1-(1+i)^{-n}}{p[(1+i)^{1/p}-1]}. \tag{20}$$

In formulas (19) and (20) the values of the annuities are expressed in terms of the effective rate i. If $(1+i)$ is replaced by $(1+j/m)^m$ in accordance with formula (2), there result the present values of the same annuities expressed as follows in terms of the nominal rate of interest j, with frequency of conversion m:

$$a_{\overline{n}|}=\frac{1-(1+j/m)^{-mn}}{(1+j/m)^m-1}, \tag{21}$$

and

$$a^{(p)}_{\overline{n}|}=\frac{1-(1+j/m)^{-mn}}{p[(1+j/m)^{m/p}-1]}. \tag{22}$$

Fundamental relations between the present value and the amount of an annuity.—Since $a_{\overline{n}|}$ and $s_{\overline{n}|}$ are the values of the same annuity upon two dates differing by n years, it follows by the principle of reduction of values from one date to another that

$$a_{\overline{n}|} = v^n s_{\overline{n}|},$$
$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|},$$

and in like manner that

$$a^{(p)}_{\overline{n}|} = v^n s^{(p)}_{\overline{n}|},$$
$$s^{(p)}_{\overline{n}|} = (1+i)^n a^{(p)}_{\overline{n}|}.$$

As tables are not published giving the values of $a^{(p)}_{\overline{n}|}$ and $s^{(p)}_{\overline{n}|}$, when p is different from 1, it is desirable for purposes of computation to express a relation between these functions and the tabulated functions $a_{\overline{n}|}$ and $s_{\overline{n}|}$. This can easily be done by accumulating to the end of each year the p payments of $1/p$ which in $a^{(p)}_{\overline{n}|}$ and $s^{(p)}_{\overline{n}|}$ are distributed at equal intervals through the year. By formula (11) this accumulation to the end of each year will be equal to

$$s^{(p)}_{\overline{1}|} = \frac{i}{p[(1+i)^{1/p}-1]} = \frac{i}{j_{(p)}}.$$

This converts the annuity into one with annual rent $s^{(p)}_{\overline{1}|}$ payable at the end of each year for n years. Therefore

$$\boldsymbol{a^{(p)}_{\overline{n}|} = s^{(p)}_{\overline{1}|}\, a_{\overline{n}|}}, \tag{23}$$

$$\boldsymbol{s^{(p)}_{\overline{n}|} = s^{(p)}_{\overline{1}|}\, s_{\overline{n}|}}. \tag{24}$$

The most frequent intervals in practice are semiannual, quarterly, and monthly, and to meet this requirement the values of $s^{(2)}_{\overline{1}|}$, $s^{(4)}_{\overline{1}|}$, and $s^{(12)}_{\overline{1}|}$ are given below for various rates of interest.

Values of $s^{(p)}_{\overline{1}|} = i/j_{(p)} = \dfrac{i}{p[(1+i)^{1/p}-1]}$

p	1½%	1¾%	2%	2¼%	2½%	2¾%	3%
2	1.00373604	1.00435603	1.00497525	1.00559371	1.00621142	1.00682837	1.00744458
4	1.00560755	1.00653878	1.00746906	1.00839839	1.00932677	1.01025422	1.01118072
12	1.00685653	1.00799571	1.00913389	1.01027107	1.01140725	1.01254243	1.01367662
p	3½%	4%	4½%	5%	5½%	6%	7%
2	1.00867475	1.00990195	1.01112621	1.01234754	1.01356596	1.01478151	1.01720402
4	1.01303094	1.01487744	1.01672026	1.01856942	1.02039495	1.02222688	1.02588002
12	1.01594203	1.01820351	1.02046109	1.02271479	1.02496465	1.02721070	1.03169143

Example 13.—What is the present value of an annuity for 30 years at effective rate 3%, payable in monthly installments of $25?

By formula (23) with $n=30$, $p=12$, $i=.03$,

$$a^{(12)}_{\overline{30}|} = s^{(12)}_{\overline{1}|} \cdot a_{\overline{30}|} = 1.01367662 \times 19.6004413 = 19.86850909.$$

Therefore the present value of a similar annuity of $25 per month, or with annual rent of $300, is

$$19.86850909 \times \$300 = \$5{,}960.55.$$

The annuity which 1 will purchase.—The present value $a_{\overline{n}|}$ of an annuity may be viewed as the principal which, invested at the effective rate of interest i, will provide a payment of 1 at the end of each year and will not be exhausted until the end of the nth year; in other words, $a_{\overline{n}|}$ is just sufficient to purchase an n year annuity of annual rent 1 payable at the end of each year. By proportion it appears that 1 will purchase an n year annuity of annual rent $1/a_{\overline{n}|}$ payable at the end of each year. This quantity may be described as the annuity which 1 will purchase, and its value is

$$\frac{1}{a_{\overline{n}|}}=\frac{i}{1-v^n}=\frac{i}{1-(1+i)^{-n}}. \tag{25}$$

This function is of great importance in annuity bond calculations, and its values are given for 60 terms and different rates of interest in Table 18.

Example 14.—To find the uniform annual payment which in 20 years will discharge a loan of \$100,000, including both principal and interest, at 5 per cent compounded annually.

In this case $n=20$, $i=.05$; employing formula (25) and referring to Table 18, it follows that a loan of 1 will be discharged, both principal and interest, by an annual payment of

$$\frac{1}{a_{\overline{20}|}}=.0802426;$$

hence the loan of \$100,000 will be likewise discharged by an annual payment of

$$.0802426\times\$100{,}000=\$8{,}024.26.$$

By similar reasoning it follows that 1 will purchase an immediate annuity-certain with annual rent $1/a_{\overline{n}|}^{(p)}$, payable in p installments each year. The value of each *periodical installment* is

$$\frac{1}{pa_{\overline{n}|}^{(p)}}=\frac{(1+j/m)^{m/p}-1}{1-(1+j/m)^{-mn}}, \tag{26}$$

where interest is at the nominal rate j with frequency of conversion m. When $m=1$, the nominal rate $j_{(1)}$ becomes the effective rate i. When the conversion of interest occurs with the same frequency as the periodical payment, that is, when $m=p$, formula (26) reduces to the important particular case

$$\frac{1}{pa_{\overline{n}|}^{(p)}}=\frac{j/p}{1-(1+j/p)^{-np}}=\frac{1}{a_{\overline{np}|}}, \tag{27}$$

where $a_{\overline{np}|}$ is to be taken at rate j/p.

Example 15.—To find the half yearly payment at 5% compounded semiannually which will discharge both principal and interest on a loan of \$100,000 in three years.

By formula (27) with $n=3$, $p=2$, a loan of 1 will be discharged, both principal and interest, in three years by a semiannual payment of

$$\frac{1}{a_{\overline{6}|}\text{ (taken at }2\tfrac{1}{2}\%)}=.1815500,$$

and the loan of \$100,000 will be discharged in like manner by

$$.1815500\times\$100{,}000=\$18{,}155.00.$$

Installment annuity loan.—The preceding example shows how the function $1/a_{\overline{n}|}$ may be employed to determine the periodical fixed payment which in n years will discharge both principal and interest on a loan. It is to be noted particularly that the lender receives interest throughout the term of the loan on all *outstanding* principal. The following schedule, based on the above example, illustrates the progress of the loan.

SCHEDULE I.—*Showing repayment of principal and interest on a loan of $100,000 by six equal semiannual payments, each of $18,155; interest 5 per cent, compounded semiannually.*

Year.	Principal outstanding at beginning of interval.	Interest for interval.	Semiannual payment.	Principal repayment for interval.
½	$100, 000. 00	$2, 500. 00	$18, 155. 00	$15, 655. 00
1	84, 345. 00	2, 108. 63	18, 155. 00	16, 046. 37
1½	68, 298. 63	1, 707. 47	18, 155. 00	16, 447. 53
2	51, 851. 10	1, 296. 28	18, 155. 00	16, 858. 72
2½	34, 992. 38	874. 81	18, 155. 00	17, 280. 19
3	17, 712. 19	442. 81	18, 155. 00	17, 712. 19
Totals	357, 199. 30	8, 930. 00	108, 930. 00	100, 000. 00

The initial invested principal of $100,000 earns $2,500 interest during the first half year; the first payment of $18,155.00 takes care of this and there remains a balance of $15,655.00 which goes to reduce the outstanding principal to $84,345.00, beginning with the second half year. This process is repeated until the end of the third year, when the last outstanding principal is retired. When preparing such a schedule, the work can be checked by adding the columns. It is evident from the nature of the calculations that, for example, if the first row were omitted from this schedule, the remaining five would represent the schedule for a loan of $84,345.00 on the same terms as the original loan, except that it would be discharged in two and one-half years by five equal semiannual payments. It must therefore be the present value of the five payments, that is,

$$a_{\overline{5}|}\times\frac{\$100{,}000}{a_{\overline{6}|}}=\$84{,}345.00,$$

where the annuities are taken at 2½ per cent. Similarly, by successively employing $a_{\overline{4}|}$, $a_{\overline{3}|}$, $a_{\overline{2}|}$, and $a_{\overline{1}|}$, all at 2½ per cent, as multipliers, the figures in the first column of principal outstanding at the beginning of the interval could be obtained. When these are known, the figures in the second column are obtained by multiplying the corresponding figures in the first column by the interest rate for the interval, .025; in the fourth, by successive subtractions of the figures

in the first; and in the third, by adding those in the second to those in the fourth as a check.

Generalization of the annuity loan.—The preceding discussion can most easily be generalized by considering the loan of $a_{\overline{n}|}$ dollars where both principal and interest at *effective rate i per annum* are discharged by equal annual installments of 1 at the end of each year for n years. The initial principal is $a_{\overline{n}|}$; the interest, $ia_{\overline{n}|}=1-v^n$; the annual payment, 1, of which $1-(1-v^n)=v^n$ is applied to repayment of principal. But $a_{\overline{n}|}-v^n=a_{\overline{n-1}|}$; hence the outstanding principal at the beginning of the second year is $a_{\overline{n-1}|}$, as might have been predicted in advance. A repetition of this process leads to the following schedule:

SCHEDULE II.—*Showing repayment of principal and interest at effective rate i per annum on a loan of $a_{\overline{n}|}$ by equal annual payments of 1 at the end of each year for n years.*

Year.	Principal outstanding at beginning of year.	Interest due at end of year.	Annual payment at end of year.	Principal repaid at end of year.
1	$a_{\overline{n}\|}$	$1-v^n$	1	v^n
2	$a_{\overline{n-1}\|}$	$1-v^{n-1}$	1	v^{n-1}
3	$a_{\overline{n-2}\|}$	$1-v^{n-2}$	1	v^{n-2}
:	:	:	:	:
k	$a_{\overline{n-k+1}\|}$	$1-v^{n-k+1}$	1	v^{n-k+1}
:	:	:	:	:
n	$a_{\overline{1}\|}$	$1-v$	1	v
Totals	$(n-a_{\overline{n}\|})/i$	$n-a_{\overline{n}\|}$	n	$a_{\overline{n}\|}$

Since this is a schedule for a loan of $a_{\overline{n}|}$, if each item in it, apart from those in the column headed "year," is divided by $a_{\overline{n}|}$ and multiplied by L, there results the corresponding schedule for a loan of L dollars.

For example, the items on a loan of L dollars for the kth year would be

$$La_{\overline{n-k+1}|}/a_{\overline{n}|},\quad L(1-v^{n-k+1})/a_{\overline{n}|},\quad L/a_{\overline{n}|},\quad Lv^{n-k+1}/a_{\overline{n}|}. \tag{28}$$

There are some curious properties revealed by the above schedule, among which the following may be pointed out. The principal repayments on an annuity loan increase in geometrical progression, the factor being $1+i$. The sum of these repayments is $a_{\overline{n}|}$; the sum of the annual payments is n; the total interest is $n-a_{\overline{n}|}$; and the check on the first and second columns shows that

$$i(a_{\overline{1}|}+a_{\overline{2}|}+\ldots\ldots\ldots+a_{\overline{n}|})=n-a_{\overline{n}|}.$$

It is apparent that most of the items in the schedule can be filled in directly from the $a_{\overline{n}|}$ and v^n tables. Having thus filled in each

number, it would be necessary only to multiply each item by $L/a_{\overline{n}|}$ to obtain the corresponding schedule for a loan of L.

If in the preceding discussion *year* is replaced by *interval,* the schedule may be made to apply to loans repaid by equal installments at the end of each interval.

Relation between annuity which 1 will purchase and sinking fund which will amount to 1.—The important relation

$$\frac{1}{a_{\overline{n}|}}=\frac{1}{s_{\overline{n}|}}+i \tag{29}$$

can easily be verified by substitution of the values of $1/a_{\overline{n}|}$ and $1/s_{\overline{n}|}$ expressed in terms of i, by formulas (25) and (15).

The relation (29) merely expresses the fact that the annual rent, $1/a_{\overline{n}|}$ on the annuity which 1 will purchase, must include, not only the interest i on the unit so invested, but also a sinking fund, $1/s_{\overline{n}|}$, which will accumulate to the invested unit at the end of the term of the annuity.

Application to bond calculations.—An important application of the theory of compound interest and annuities arises in the valuation of bonds. First to determine the value of a bond issue redeemable in one sum on a given date, with interest, or dividends, on the outstanding bonds at rate g, and all computed, or *valued,* so as to yield the purchaser a given effective rate of interest i. Consider an issue of $100,000 highway bonds, denomination $500, dated January 1, 1914, maturing January 1, 1948, interest 5 per cent, payable annually.

The annual interest, or dividends, on these bonds is 5 per cent, and the bonds are redeemed at the end of 34 years. Suppose an intending purchaser desires to pay a price which will yield a net income of 3 per cent on his investment; how much ought he to bid? This is the nature of the general problem. If the purchaser desires to realize 5 per cent on his investment, he must bid $100,000 for the bonds, or $1 for each dollar to be redeemed. If, however, he is content with 3 per cent, more than $100,000 must be paid for the bonds, that is, more than $1 for each dollar to be redeemed. In this case the bonds are said to be bought at a *premium;* if less than $1 is paid for each dollar to be redeemed, the bonds are said to be bought at a *discount.*

In the general case, let C denote the price to be paid on redemption; i, the effective rate of interest employed in the valuation of the bonds, which is the *net income* rate to the purchaser; g, the *ratio* of the dividend per annum to C; n, the number of years after which the bonds are redeemed; K, the present value of C, due n years hence,

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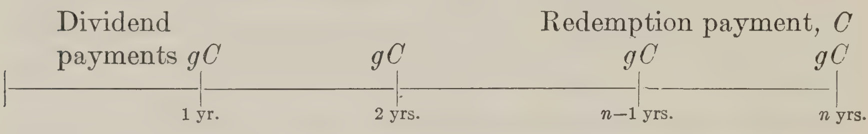
at the effective rate of interest i; and A, the present value of, or bid on, the bonds.

In the above illustration $C = 100{,}000$, and $n = 34$. The dividend or interest per annum is 5,000. Hence $g = 5{,}000/100{,}000 = .05$.

Returning to the general problem, the value of the bonds, so far as the purchaser or holder is concerned, consists of two parts:

1. *The annual interest, or dividend, to be received.*
2. *The sum to be redeemed at the end of n years.*

Hence, to find the present value, A, of the bonds, the present value of each of these parts must be determined and added together. The interest per unit of the redemption price C is, by definition, g; if the interest on 1 unit is g, the interest on C units is gC. Hence at the end of every year for n years the holder will receive gC units.



It is evident that these interest or dividend payments of gC at the end of every year constitute an immediate annuity-certain of annual rent gC and term of n years. The value of such an annuity with annual rent 1 is $a_{\overline{n}|}$; hence the value of the annuity with annual rent gC is

$$gC\, a_{\overline{n}|},$$

where $a_{\overline{n}|}$ is to be taken at the rate of interest i *to be employed in the valuation* of the bonds, a rate which in general is different from g, the rate of dividend.

By formula (5), the present value of the sum C, to be redeemed in n years, is $v^n{=}C$.

Adding these parts together, the result is

$$A = v^n C + gCa_{\overline{n}|}.$$

Substituting in this relation the value of $a_{\overline{n}|}$ given by formula (19), it follows that

$$A = v^n C + \frac{g}{i}(C - v^n C).$$

Since, by definition, $K = v^n C$, the *bid* is given by

$$\boldsymbol{A = K + \frac{g}{i}(C - K)} \tag{30}$$

and the *premium* by

$$\boldsymbol{A - C = (C - K)\frac{(g - i)}{i}}. \tag{31}$$

If in formula (31) the *total sum to be redeemed is regarded as unity,* then $C=1$ and $K=v^n$, the present value of 1 due in n years, and there results

$$A=1+\frac{(1-v^n)}{i}(g-i)=1+(g-i)a_{\overline{n}|}. \tag{32}$$

In this formula $a_{\overline{n}|}$ is taken at i per cent, and gives the bid on a bond where the sum to be redeemed is 1. Denoting the excess of A over 1 by k, which is called the *premium,* formula (32) becomes

$$k=(g-i)a_{\overline{n}|}^{i\%}, \tag{33}$$

where the i per cent over the symbol $a_{\overline{n}|}$ means that the function is to be taken from the i per cent annuity table.

This is the fundamental formula in bond calculations. It admits of a very simple interpretation, for it states that the premium on a bond is equal to the present value of an n year annuity at i per cent whose annual rent is the excess $(g-i)$ of the nominal rate of dividend of the bond over the effective rate of interest i, desired to be realized by the investor.

				Unit redeemed, 1	
Unit invested, 1		i	i	i	i
		1 yr.	2 yrs.	$n-1$ yrs.	n yrs.
Premium,	k	$g-i$	$g-i$	$g-i$	$g-i$

The dividend paid each year on each unit of the bond to be redeemed is g, which may be divided into two parts, i and $g-i$. For the first part the investor pays 1 and in return receives interest of i each year and the 1 is redeemed at the end of n years. For the second part the investor pays the premium, $k=(g-i)a_{\overline{n}|}$, and this is repaid, both principal and interest at rate i, in n equal annual installments of $(g-i)$. A portion of each installment goes toward the repayment of the premium k which is eventually reduced to zero. This is called the *amortization* or *writing off of the premium.*

It is thus seen that, if k is positive, the bond is bought at a premium; and if k is negative, it is bought at a discount. Since $a_{\overline{n}|}$ is always positive, it appears from formula (33) that the sign of k will be positive when g is greater than i, or *when the rate of dividend is greater than the rate of interest used in valuation;* conversely, when g is less than i, k is negative.

Example 16.—To find the bid on the highway bond mentioned on page 25, on the hypothesis that the purchaser wishes to realize 3% on his investment.

Consider a dollar (unit) of the loan $C=100{,}000$. Here $n=34$, $g=.05$, $i=.03$, and by formula (33),

$$k=(.05-.03)a_{\overline{34}|}^{3\%}=.02\times 21.1318367=.422636734,$$

or the premium is slightly over 42 cents on the dollar. Since for each dollar of the loan the purchaser must pay \$1.422636734, for the whole loan of \$100,000 he must pay

$$1.422636734\times\$100{,}000=\$142{,}263.67.$$

Dividends payable and interest convertible semiannually.—When the net income interest rate desired by the investor is nominal, say $j_{(m)}$, and the dividends per unit of the sum to be redeemed are paid in m equal installments, g/m, during the year, it is evident that it is a case of m times n intervals with g/m as dividend and j/m as the *effective rate of interest per interval.* Hence formula (33) becomes

$$k=\frac{(g-j)}{m}a_{\overline{mn}|}^{j/m\%}. \tag{34}$$

In particular, if the net income is $j_{(2)}$, and the dividend payments are semiannual,

$$k=\frac{(g-j)}{2}a_{\overline{2n}|}^{j/2\%}. \tag{35}$$

This formula provides for the valuation of all bonds, redeemed in one sum at the end of a term of n years and with semiannual dividends. Particular attention is called to the fact that the annuity must be taken for the term $2n$, and at the rate of interest $j/2$.

Example 17.—What is the bid on \$100,000 highway 5% bonds maturing at the end of 3 years, interest payable semiannually, to net purchaser a nominal rate of 4% convertible half-yearly?

Here $n=3$, $g=.05$, $j=.04$, $m=2$, and formula (35) gives

$$k=\frac{(.05-.04)}{2}a_{\overline{6}|}^{2\%}=.005\times5.6014309=.0280071545.$$

Hence the premium on \$100,000 is \$2,800.72, and the corresponding bid is \$102,800.72. The progress of this bond loan is illustrated in the following schedule.

SCHEDULE III.

Year.	Book value or principal at beginning of half-year.	Semiannual interest of 2%.	Semiannual dividend of 2½% on bonds.	Amortization of premium at end of half-year.	Redemption payment at end of half-year.
½	\$102, 800. 72	\$2, 056. 01	\$2, 500. 00	\$443. 99	0. 00
1	102, 356. 73	2, 047. 13	2, 500. 00	452. 87	0. 00
1½	101, 903. 86	2, 038. 08	2, 500. 00	461. 92	0. 00
2	101, 441. 94	2, 028. 84	2, 500. 00	471. 16	0. 00
2½	100, 970. 78	2, 019. 42	2, 500. 00	480. 58	0. 00
3	100, 490. 20	2, 009. 80	2, 500. 00	490. 20	\$100, 000. 00
Totals	609, 964. 23	12, 199. 28	15, 000. 00	2, 800. 72	100, 000. 00

At the outset the holder has an investment of \$102,800.72 upon which, at 2 per cent, at the end of the first half-year, \$2,056.01 interest is due; the dividend payment of \$2,500.00 then made on the bonds provides for this interest and a balance of \$443.99 remains, which is applied to *amortize* or *write off* the premium so that the *book-value*, or invested principal, is reduced to \$102,356.73 at the beginning of the second half-year. This process continues for three years until the entire premium of \$2,800.72 is written off and the bonds are redeemed by the payment of \$100,000. The various columns are added and the checks upon these totals are obvious.

Example 18.—What is the bid on \$100,000 highway 3% bonds maturing at the end of 3 years, interest payable semiannually, to net purchaser a nominal rate of 4% convertible half-yearly?

Here $n=3$, $g=.03$, $j=.04$, $m=2$, and formula (35) gives

$$k=\frac{(.03-.04)}{2}a_{\overline{6}|}^{2\%}=-.005\times5.6014309=-.0280071545.$$

Hence the *discount* on \$100,000 is \$2,800.72, and the corresponding bid is \$97,199.28. The progress of this bond loan is illustrated in the following schedule.

SCHEDULE IV.

Year.	Book value or principal at beginning of half-year.	Semiannual interest of 2%.	Semiannual dividend of 1½% on bonds.	Accumulation of discount at end of half-year.	Redemption payment at end of half-year.
½	\$97, 199. 28	\$1, 943. 99	\$1, 500. 00	\$443. 99	0. 00
1	97, 643. 27	1, 952. 87	1, 500. 00	452. 87	0. 00
1½	98, 096. 14	1, 961. 92	1, 500. 00	461. 92	0. 00
2	98, 558. 06	1, 971. 16	1, 500. 00	471. 16	0. 00
2½	99, 029. 22	1, 980. 58	1, 500. 00	480. 58	0. 00
3	99, 509. 80	1, 990. 20	1, 500. 00	490. 20	\$100, 000. 00
Totals	590, 035. 77	11, 800. 72	9, 000. 00	2, 800. 72	100, 000. 00

In this case the holder has an initial investment of \$97,199.28, and at the end of the first half-year 2 per cent interest, or \$1,943.99, is due. The dividend payment of \$1,500.00, then made on the bonds, is *not sufficient* to provide for this interest, and the difference of \$443.99 is added to the principal and determines the *book value* at the beginning of the second half-year. This is called the *accumulation* or *writing on* of discount. By continuing this process for three years the entire discount of \$2,800.72 is written on the initial principal, and the book value, \$100,000, is then redeemed. The totals of the several columns may be used to check the numerical work.

Valuation of bonds redeemed in installments.—For the valuation of bonds which are not redeemed in one sum, but in a series of installments, first consider the simpler case where the dividend payments are annual and the rate of interest is the effective rate i.

Let $C_1, C_2, \ldots\ldots C_r$, denote the successive installments by which the bonds are to be redeemed;

$n_1, n_2, \ldots\ldots n_r$, the respective number of years after which the successive installments become due;

$K_1, K_2, \ldots\ldots K_r$, the present values, at the effective rate of interest i, of

C_1 due n_1 years hence,

C_2 due n_2 years hence,

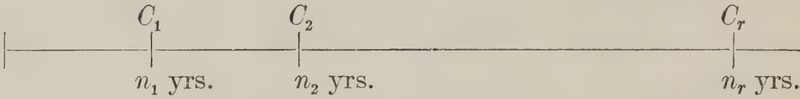
. . . .

C_r due n_r years hence;

g, the fixed rate of dividend to be paid on the *outstanding* bonds;

i, the effective rate of interest employed in the valuation of the bonds, which is the *net income* rate to the purchaser;

and $A_1, A_2, \ldots \ldots A_r$, the present values, at the effective rate i, of the separate installments *with their respective dividends.*



Each installment redeemed may be regarded as furnishing a distinct problem under formula (30) so that, in order to value the entire bond issue, it may be treated as made up of r distinct issues and, after finding the value of each one, they may be added together for the value or bid on the total issue.

By formula (30) in the case of a single issue of C_1 at *net income* rate i, dividend rate g, due in n_1 years, the present value, or bid, A_1, is:

$$A_1 = K_1 + (g/i)\ (C_1 - K_1).$$

Similarly,

$$A_2 = K_2 + (g/i)\ (C_2 - K_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$A_r = K_r + (g/i)\ (C_r - K_r).$$

Adding,

$$(A_1 + A_2 + \ldots + A_r) = (K_1 + K_2 + \ldots + K_r)$$
$$+ (g/i)[(C_1 + C_2 + \ldots + C_r) - (K_1 + K_2 + \ldots + K_r)].$$

The total sum to be redeemed, $C_1 + C_2 + \ldots + C_r$, is denoted by C; the total present value of C_1 in n_1 years, C_2 in n_2 years, and so on, which by definition is equal to $K_1 + K_2 + \ldots + K_r$, by K; and the total value of the issue, $A_1 + A_2 + \ldots + A_r$, by A; then for the bid there results

$$\boldsymbol{A = K + (g/i)\ (C - K)}, \tag{36}$$

and for the premium,

$$\boldsymbol{A - C = (C - K)\ (g - i)/i}. \tag{37}$$

It thus appears that formulas (30) and (31), which were derived before for the case of a bond issue redeemed in one sum, hold also for the more general issue redeemed in any number of installments.

Installment bonds when total sum to be redeemed is 1.— When 1 is the total sum to be redeemed, that is, when $C = 1$, formula (37) becomes

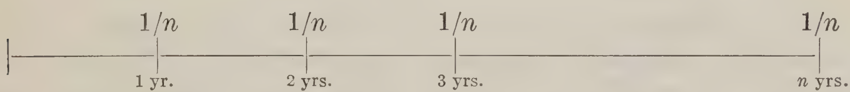
$$\boldsymbol{A - 1 = (1 - K)\ (g - i)/i}, \tag{38}$$

where A is the value of each unit of the sum to be redeemed, and K is the present value of the various parts of the unit at effective rate i due in $n_1, n_2, \ldots\ldots n_r$ years. Letting $A-1=k$, formula (38) becomes

$$\boldsymbol{k}=(1-\boldsymbol{K})\ (\boldsymbol{g}-\boldsymbol{i})/\boldsymbol{i}. \tag{39}$$

The premium is positive if g is greater than i, and negative, or a discount, if g is less than i; for the first factor $(1-K)$ can not be negative, as K by definition is the *present value* of a series of future payments whose sum is 1, and hence their present discounted value must be less than 1. This shows in all cases that a bond issue must be bought at a *premium*, if it is valued at a *lower* rate i than the rate of dividend g; and at a discount, if it is valued at a higher rate i than the rate of dividend g.

Serial bonds.—To apply the general formula (39) to the case of a bond issue redeemed by n equal annual installments, consider a unit of the total sum to be redeemed. Since this unit is to be redeemed in n equal installments over n years, the annual portion redeemed is $1/n$.



The present value, K, of these n installments is clearly the value of an annuity of annual rent $1/n$; hence

$$K=a_{\overline{n}|}\times 1/n=a_{\overline{n}|}/n.$$

Substituting this value of K in formula (39), the following formula is obtained:

$$\boldsymbol{k}=(1-\boldsymbol{a}_{\overline{n}|}/\boldsymbol{n})\ (\boldsymbol{g}-\boldsymbol{i})/\boldsymbol{i}. \tag{40}$$

Example 19.—What is the bid on \$100,000 highway 4% serial bonds maturing in 20 equal annual installments, to net the purchaser an effective rate of 3%?

Here $n=20$, $g=.04$, $i=.03$, and $a_{\overline{20}|}^{3\%}=14.8774749$; consequently

$$k=(1-14.8774749/20)(.04-.03)/.03$$
$$=(1-.743873745)\times 1/3=.256126255\times 1/3=.085375418.$$

Hence the bid on \$100,000 is

$$1.085375418\times\$100{,}000=\$108{,}537.54.$$

Extension of formulas to case when dividends are payable and interest is convertible *m* times per annum.—Formula (36) assumes that dividends are payable once a year and that the effective rate of interest is i per annum. Replacing *year* by *interval* and assuming dividends to be paid at the end of each interval and the rate of interest realized by the investor a nominal rate convertible m times a year, formula (36) still applies, if the present value K of the several

installments to be redeemed is calculated at the effective rate j/m per interval, and the dividend per unit of the sum to be redeemed is taken at the rate g/m per interval. The formula is unchanged in form since m cancels out in the ratio g/m to j/m.

General formula for valuation of bonds.—Assume that:

1. The bonds are redeemed in r equal installments.
2. The first redemption of bonds is made at the end of f years.
3. The remaining $r-1$ bond redemptions are made at intervals of t years.
4. The annual rate of dividend is g paid in m equal installments.
5. The bond issue is valued at the nominal rate $j_{(m)}$.

First find the present value, A, of an issue of the above type where $C=1$. The value of a similar total issue of C is then found by multiplying A by C. Since the unit fund is redeemed in r equal installments, each one will be $1/r$.

Redemption payments $\quad 1/r \quad 1/r \quad 1/r \quad 1/r$

f years $\quad t$ yrs. $\quad t$ yrs.

The total term of the issue is seen to be $f+(r-1)t$ years. As in preceding extension of formulas when dividends are payable and interest is convertible m times per annum, apply formula (36) to each installment of $1/r$ in the unit issue and the formula for the value of k, the premium per unit of the total sum to be redeemed, may readily be obtained. Expressed in terms of annuities, it appears as follows:

$$k=\left[1-\frac{a_{\overline{m(f+tr)}|}-a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right](g-j)/j \qquad \text{at rate } j/m. \qquad (41)$$

The annuity present values in this formula must be computed at the rate of interest j/m. The most common case in practice is where the dividends are paid semiannually. Here $m=2$, and formula (41) becomes:

$$k=\left[1-\frac{a_{\overline{2(f+tr)}|}-a_{\overline{2f}|}}{r a_{\overline{2t}|}}\right](g-j)/j \qquad \text{at rate } j/2. \qquad (42)$$

The last two formulas are very general in their application and have the advantage that when employed in practical computations it is necessary to consult only a table of values of $a_{\overline{n}|}$.

Example 20.—To find the bid on \$1,100,000 highway bonds, interest 5% payable semiannually, dated January 1, 1914, maturing \$100,000 on January 1, 1922, 1924, 1926, 1928, 1930, 1932, 1934, 1936, 1938, 1940, and 1942, to net the purchaser a nominal rate of 4%, compounded semiannually, on his investment.

Here $f=8$, $t=2$, $r=11$, $g=.05$, $m=2$, and $j_{(2)}=.04$. Accordingly, $m(f+tr)=60$, $mf=16$, and $mt=4$. Substituting in formula (42),

$$k=\left[1-\frac{a_{\overline{60}|}-a_{\overline{16}|}}{11\times a_{\overline{4}|}}\right](.05-.04)/.04 \qquad \text{at } 2\%.$$

Entering Table 17 with 2% for the values of the annuities and numbering the successive steps for convenience of explanation, the calculation may be outlined as follows:

$$a_{\overline{60}|} = 34.7608867 \quad (1)$$
$$a_{\overline{16}|} = 13.5777093 \quad (2)$$
$$a_{\overline{60}|} - a_{\overline{16}|} = 21.1831774 \quad (3)$$
$$(3) \div 11 = 1.9257434 \quad (4)$$
$$a_{\overline{4}|} = 3.8077287$$
$$(4) \div a_{\overline{4}|} = .5057460 \quad (5)$$
$$\text{Complement of } (5) = 1 - (5) = .4942540 \quad (6) = \text{first factor}$$
$$(.05 - .04)/.04 = .25 \quad (7) = \text{second factor}$$
$$k = (6) \times (7) = .1235635.$$

The bid on one dollar is $1+k=1.1235635$; consequently the bid on the whole issue is

$$1.1235635 \times \$1,100,000 = \$1,235,919.85.$$

Example 21.—To find the price of $100,000 highway bonds, interest 5%, semiannual, dated January 1, 1914, maturing $50,000 January 1, 1917, and $50,000 January 1, 1919, to net the investor 4% compounded semiannually.

In this case $f=3$, $r=2$, $t=2$, $m=2$, $g=.05$, $j=.04$, and, substituting as in the preceding example, the required price is found to be $103,646.00. The progress of the loan is indicated in the following schedule.

SCHEDULE V.

Year.	Book value or principal at beginning of half-year.	Semiannual interest of 2%.	Semiannual dividend of 2½% on bonds.	Amortization of premium at end of half-year.	Redemption payment at end of half-year.
½	$103, 646. 00	$2, 072. 92	$2, 500. 00	$427. 08	0. 00
1	103, 218. 92	2, 064. 38	2, 500. 00	435. 62	0. 00
1½	102, 783. 30	2, 055. 67	2, 500. 00	444. 33	0. 00
2	102, 338. 97	2, 046. 78	2, 500. 00	453. 22	0. 00
2½	101, 885. 75	2, 037. 72	2, 500. 00	462. 28	0. 00
3	101, 423. 47	2, 028. 47	2, 500. 00	471. 53	$50, 000. 00
3½	50, 951. 94	1, 019. 04	1, 250. 00	230. 96	0. 00
4	50, 720. 98	1, 014. 42	1, 250. 00	235. 58	0. 00
4½	50, 485. 40	1, 009. 71	1, 250. 00	240. 29	0. 00
5	50, 245. 11	1, 004. 89	1, 250. 00	245. 11	50, 000. 00
Totals	817, 699. 84	16, 354. 00	20, 000. 00	3, 646. 00	100, 000. 00

Extension of term of tables.—It sometimes happens in applying formula (42) that the value of $2(f+tr)$ is greater then the term given in the tables. In example 20 one of the required annuity values was $a_{\overline{60}|}$ but, if the interval between redemptions had been three years instead of two, $2(f+tr)=82$ would have called for the value of an annuity $a_{\overline{82}|}$ beyond the limits of the tables. It is easy, however, to extend these limits by making use of the following obvious relations:

$$v^{m+n} = v^m v^n, \tag{43}$$

$$(1+i)^{m+n} = (1+i)^m (1+i)^n, \tag{44}$$

$$a_{\overline{m+n}|} = [1 - v^m v^n]/i, \tag{45}$$

$$a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|}, \tag{46}$$

$$s_{\overline{m+n}|} = [(1+i)^m (1+i)^n - 1]/i, \tag{47}$$

$$s_{\overline{m+n}|} = (1+i)^n s_{\overline{m}|} + s_{\overline{n}|}. \tag{48}$$

Example 22.—To find $s_{\overline{94}|}$ at $1\frac{1}{2}\%$ when the limit of the tables is 60 years or terms.

Applying formula (47) there results

$$s_{\overline{94}|}=s_{\overline{60+34}|}=\frac{(1.015)^{60+34}-1}{.015}=\frac{(1.015)^{60}\times(1.015)^{34}-1}{.015}$$

$$=\frac{2.4432198\times1.6589964-1}{.015}=203.5528568.$$

By formula (48)

$$s_{\overline{94}|}=s_{\overline{60+34}|}=(1.015)^{34}\cdot s_{\overline{60}|}+s_{\overline{34}|}$$

$$=1.6589964\times96.2146517+43.9330915=203.5528523.$$

The correct value of $s_{\overline{94}|}$ at $1\frac{1}{2}\%$ to seven places of decimals is 203.5528497; so the above method may be regarded as giving the correct value to about five places of decimals. In most practical cases this will be sufficiently accurate.

Valuation of serial bonds bearing semiannual dividends.—The most common type of serial bond bears semiannual dividends and is redeemed in equal *annual* installments, the first of which is paid at the end of the first year. Formula (42) lends itself directly to the valuation of this bond at a nominal rate of interest j convertible twice a year. In this case $f=t=1$, $r=n$, and

$$\boldsymbol{k}=\left[1-\frac{a_{\overline{2n+2}|}-a_{\overline{2}|}}{n a_{\overline{2}|}}\right](g-j)/j \qquad \text{at rate } j/2. \tag{49}$$

Formula (49) requires the use of a table of values of $a_{\overline{n}|}$ only. It can be put in another convenient form for computation involving the use of a table of values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$. For, by formula (46), $a_{\overline{2+2n}|}=a_{\overline{2}|}+v^2a_{\overline{2n}|}$, and, since $v^2/a_{\overline{2}|}=1/(1+i)^2a_{\overline{2}|}=1/s_{\overline{2}|}$, after a simple reduction, there results

$$\boldsymbol{k}=\left[1-\frac{a_{\overline{2n}|}}{n s_{\overline{2}|}}\right](g-j)/j \qquad \text{at rate } j/2. \tag{50}$$

Example 23.—\$300,000 highway serial bonds bearing 4% interest payable semiannually, dated January 1, 1914, mature \$100,000 January 1, 1915, 1916, and 1917. What price should be paid to realize a net income of 3% compounded semiannually?

Here $n=3$, $g=.04$, $j_{(2)}=.03$, and by formula (49)

$$k=\left[1-\frac{a_{\overline{8}|}-a_{\overline{2}|}}{3a_{\overline{2}|}}\right](.04-.03)/.03 \qquad \text{at } 1\tfrac{1}{2}\%$$

$$=.0575373\times1/3=.0191791,$$

therefore the price to earn 3% compounded semiannually is

$$1.0191791\times\$300{,}000=\$305{,}753.73.$$

The following schedule illustrates the progress of this loan.

SCHEDULE VI.

Year.	Book value or principal at beginning of half-year.	Semiannual interest of $1\frac{1}{2}\%$.	Semiannual dividend of 2% on bonds.	Amortization of premium at end of half-year.	Redemption payment at end of half-year.
$\frac{1}{2}$	$305, 753.73	$4, 586.31	$6, 000.00	$1, 413.69	0.00
1	304, 340.04	4, 565.10	6, 000.00	1, 434.90	$100, 000.00
$1\frac{1}{2}$	202, 905.14	3, 043.58	4, 000.00	956.42	0.00
2	201, 948.72	3, 029.23	4, 000.00	970.77	100, 000.00
$2\frac{1}{2}$	100, 977.95	1, 514.67	2, 000.00	485.33	0.00
3	100, 492.62	1, 507.38	2, 000.00	492.62	100, 000.00
Totals	1, 216, 418.20	18, 246.27	24, 000.00	5, 753.73	300, 000.00

Annuity bonds.—On pages 22 to 25 the operation of a loan where both principal and interest are discharged by equal installments is fully described. It is evident that bonds may be issued on this basis and retired in accordance with the principal repayments contained in the annuity installments. Since these principal repayments are not exact multiples of the amounts or denominations in which bonds are usually issued, it is necessary to adjust the *exact* schedule so as to meet this requirement. The adjusted schedule gives an issue in which the bonds are retired year by year in increasing amounts. Examples of exact and adjusted schedules appear in the body of this bulletin on pages 5 and 6.

To finance a loan of L by an issue of annuity bonds bearing interest or dividends at rate g per annum.—The annual installment which will retire the bonds in n years and at the same time pay interest at the rate of g per cent on outstanding bonds is

$$L/a_{\overline{n}|} \qquad \text{at rate } g. \tag{51}$$

If the bonds are to bear interest of g per cent per annum, payable in p installments of g/p per cent during the year, then

$$L/a_{\overline{np}|} \qquad \text{at rate } g/p \tag{52}$$

is the periodical payment or annuity installment which will take care of interest on the bonds and retire them in n years.

Example 24.—Adjust Schedule I, page 23, to finance the same loan by an annuity bond issue of $100,000, denomination $100, bearing 5% interest, compounded semiannually, and retired in three years by six equal (nearly) semiannual annuity installments.

Referring to Schedule I on page 23, the adjustments in the last column to even multiples of $100 are easily made; a check on this work is that the adjusted column must foot up to $100,000. When the column of bond redemptions is decided upon the other columns in the schedule are readily derived.

Schedule VII.

(Schedule I adjusted to bonds of denomination $100.)

Year.	Book value or principal at beginning of half-year.	Semiannual interest of $2\frac{1}{2}\%$.	Annuity installments at end of half-year.	Amortization of premium at end of half-year.	Amount of bonds retired at end of half-year.
$\frac{1}{2}$	$100, 000	$2, 500. 00	$18, 200. 00	0. 00	$15, 700
1	84, 300	2, 107. 50	18, 107. 50	0. 00	16, 000
$1\frac{1}{2}$	68, 300	1, 707. 50	18, 107. 50	0. 00	16, 400
2	51, 900	1, 297. 50	18, 197. 50	0. 00	16, 900
$2\frac{1}{2}$	35, 000	875. 00	18, 175. 00	0. 00	17, 300
3	17, 700	442. 50	18, 142. 50	0. 00	17, 700
Totals	357, 200	8, 930. 00	108, 930. 00	0. 00	100, 000

Valuation of annuity bonds.—In order to value an issue of this character, so as to yield the purchaser a net income at a rate of interest different from the rate of dividend on the bonds, it will ordinarily be necessary to value separately the several parts of the total issue in accordance with the respective dates on which they are retired. This calculation may frequently be shortened by employing formula (36). Bond tables may also be consulted to advantage. The following example and schedule respectively illustrate the calculation of the bid and progress of the loan.

Example 25.—Determine the bid on the entire issue of annuity bonds in Example 24 so as to yield the investor a net income of 4%, compounded semiannually.

Applying formula (35) successively to the several bond issues in the order in which they are retired with $g=.05$ and $j=.04$, the following premiums are found:

$76. 96
155. 32
236. 48
321. 75
407. 71
495. 73

$1, 693. 95

Accordingly, the bid on the entire issue is $101,693.95. The schedule illustrating the progress of this bond issue follows. It is constructed in the same manner as preceding bond schedules and needs no additional explanation.

Schedule VIII.—*Showing the progress of an annuity bond issue of $100,000, denomination $100, bearing 5 per cent interest, compounded semiannually, and retired in three years by six equal (nearly) semiannual annuity installments. Bought to yield the investor 4 per cent, compounded semiannually.*

Year.	Book value or principal at beginning of half-year.	Semiannual interest of 2%.	Annuity installments at end of half-year.	Amortization of premium at end of half-year.	Amount of bonds retired at end of half-year.
½	$101, 693. 95	$2, 033. 88	$18, 200. 00	$466. 12	$15, 700
1	85, 527. 83	1, 710. 56	18, 107. 50	396. 94	16. 000
1½	69, 130. 89	1, 382. 62	18, 107. 50	324. 88	16, 400
2	52, 406. 01	1, 048. 12	18, 197. 50	249. 38	16, 900
2½	35, 256. 63	705. 13	18, 175. 00	169. 87	17, 300
3	17, 786. 76	355. 74	18, 142. 50	86. 76	17, 700
Totals	361, 802. 07	7, 236. 05	108, 930. 00	1, 693. 95	100, 000

Table 13.—*The accumulation of 1 at the end of n years.*

$$r^n = (1+i)^n.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	1.0150000	1.0175000	1.0200000	1.0225000	1.0250000	1.0275000	1.0300000	1
2	1.0302250	1.0353063	1.0404000	1.0455063	1.0506250	1.0557563	1.0609000	2
3	1.0456784	1.0534241	1.0612080	1.0690301	1.0768906	1.0847896	1.0927270	3
4	1.0613636	1.0718590	1.0824322	1.0930833	1.1038129	1.1146213	1.1255088	4
5	1.0772840	1.0906166	1.1040808	1.1176777	1.1314082	1.1452733	1.1592741	5
6	1.0934433	1.1097024	1.1261624	1.1428254	1.1596934	1.1767684	1.1940523	6
7	1.1098449	1.1291222	1.1486857	1.1685390	1.1886858	1.2091295	1.2298739	7
8	1.1264926	1.1488818	1.1716594	1.1948311	1.2184029	1.2423806	1.2667701	8
9	1.1433900	1.1689872	1.1950926	1.2217148	1.2488630	1.2765460	1.3047732	9
10	1.1605408	1.1894445	1.2189944	1.2492034	1.2800845	1.3116510	1.3439164	10
11	1.1779489	1.2102598	1.2433743	1.2773105	1.3120867	1.3477214	1.3842339	11
12	1.1956182	1.2314393	1.2682418	1.3060500	1.3448888	1.3847838	1.4257609	12
13	1.2135524	1.2529895	1.2936066	1.3354361	1.3785110	1.4228653	1.4685337	13
14	1.2317557	1.2749168	1.3194788	1.3654834	1.4129738	1.4619941	1.5125897	14
15	1.2502321	1.2972279	1.3458683	1.3962068	1.4482982	1.5021990	1.5579674	15
16	1.2689856	1.3199294	1.3727857	1.4276215	1.4845056	1.5435094	1.6047064	16
17	1.2880203	1.3430281	1.4002414	1.4597429	1.5216183	1.5859560	1.6528476	17
18	1.3073406	1.3665311	1.4282463	1.4925872	1.5596587	1.6295697	1.7024331	18
19	1.3269508	1.3904454	1.4568112	1.5261704	1.5986502	1.6743829	1.7535061	19
20	1.3468550	1.4147782	1.4859474	1.5605092	1.6386164	1.7204284	1.8061112	20
21	1.3670578	1.4395368	1.5156663	1.5956207	1.6795819	1.7677402	1.8602946	21
22	1.3875637	1.4647287	1.5459797	1.6315221	1.7215714	1.8163531	1.9161034	22
23	1.4083772	1.4903615	1.5768993	1.6682314	1.7646107	1.8663028	1.9735865	23
24	1.4295028	1.5164428	1.6084373	1.7057666	1.8087260	1.9176261	2.0327941	24
25	1.4509454	1.5429805	1.6406060	1.7441463	1.8539441	1.9703608	2.0937779	25
26	1.4727095	1.5699827	1.6734181	1.7833896	1.9002927	2.0245458	2.1565913	26
27	1.4948002	1.5974574	1.7068865	1.8235159	1.9478000	2.0802208	2.2212890	27
28	1.5172222	1.6254129	1.7410242	1.8645450	1.9964950	2.1374268	2.2879277	28
29	1.5399805	1.6538576	1.7758447	1.9064973	2.0464074	2.1962061	2.3565655	29
30	1.5630802	1.6828001	1.8113616	1.9493934	2.0975676	2.2566017	2.4272625	30
31	1.5865264	1.7122491	1.8475888	1.9932548	2.1500068	2.3186583	2.5000804	31
32	1.6103243	1.7422135	1.8845406	2.0381030	2.2037569	2.3824214	2.5750828	32
33	1.6344792	1.7727022	1.9222314	2.0839603	2.2588509	2.4479380	2.6523352	33
34	1.6589964	1.8037245	1.9606760	2.1308495	2.3153221	2.5152563	2.7319053	34
35	1.6838813	1.8352897	1.9998896	2.1787936	2.3732052	2.5844258	2.8138625	35
36	1.7091395	1.8674073	2.0398873	2.2278164	2.4325353	2.6554975	2.8982783	36
37	1.7347766	1.9000869	2.0806851	2.2779423	2.4933487	2.7285237	2.9852267	37
38	1.7607983	1.9333384	2.1222988	2.3291960	2.5556824	2.8035581	3.0747835	38
39	1.7872103	1.9671718	2.1647448	2.3816029	2.6195745	2.8806560	3.1670270	39
40	1.8140184	2.0015973	2.2080397	2.4351890	2.6850638	2.9598740	3.2620378	40
41	1.8412287	2.0366253	2.2522005	2.4899807	2.7521904	3.0412705	3.3598989	41
42	1.8688471	2.0722662	2.2972445	2.5460053	2.8209952	3.1249055	3.4606959	42
43	1.8968798	2.1085309	2.3431894	2.6032904	2.8915201	3.2108404	3.5645168	43
44	1.9253330	2.1454302	2.3900531	2.6618644	2.9638081	3.2991385	3.6714523	44
45	1.9542130	2.1829752	2.4378542	2.7217564	3.0379033	3.3898648	3.7815958	45
46	1.9835262	2.2211773	2.4866113	2.7829959	3.1138509	3.4830861	3.8950437	46
47	2.0132791	2.2600479	2.5363435	2.8456133	3.1916971	3.5788709	4.0118950	47
48	2.0434783	2.2995987	2.5870704	2.9096396	3.2714896	3.6772899	4.1322519	48
49	2.0741305	2.3398417	2.6388118	2.9751065	3.3532768	3.7784154	4.2562194	49
50	2.1052424	2.3807889	2.6915880	3.0420464	3.4371087	3.8823218	4.3839060	50
51	2.1368211	2.4224527	2.7454198	3.1104924	3.5230364	3.9890856	4.5154232	51
52	2.1688734	2.4648457	2.8003282	3.1804785	3.6111124	4.0987855	4.6508859	52
53	2.2014065	2.5079805	2.8563348	3.2520393	3.7013902	4.2115021	4.7904125	53
54	2.2344276	2.5518701	2.9134614	3.3252102	3.7939249	4.3273184	4.9341249	54
55	2.2679440	2.5965279	2.9717307	3.4000274	3.8887730	4.4463196	5.0821486	55
56	2.3019631	2.6419671	3.0311653	3.4765280	3.9859924	4.5685934	5.2346131	56
57	2.3364926	2.6882015	3.0917886	3.5547499	4.0856422	4.6942298	5.3916514	57
58	2.3715400	2.7352450	3.1566244	3.6347318	4.1877832	4.8233211	5.5534010	58
59	2.4071131	2.7831118	3.2166969	3.7165132	4.2924778	4.9559624	5.7200030	59
60	2.4432198	2.8318163	3.2810308	3.8001348	4.3997897	5.0922514	5.8916031	60

TABLE 13.—*The accumulation of 1 at the end of n years*—Continued.

$$r^n=(1+i)^n.$$

Years.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Years.
1	1.0350000	1.0400000	1.0450000	1.0500000	1.0550000	1.0600000	1.0700000	1
2	1.0712250	1.0816000	1.0920250	1.1025000	1.1130250	1.1236000	1.1449000	2
3	1.1087179	1.1248640	1.1411661	1.1576250	1.1742414	1.1910160	1.2250430	3
4	1.1475230	1.1698586	1.1925186	1.2155063	1.2388247	1.2624770	1.3107960	4
5	1.1876863	1.2166529	1.2461819	1.2762816	1.3069600	1.3382256	1.4025517	5
6	1.2292553	1.2653190	1.3022601	1.3400956	1.3788428	1.4185191	1.5007304	6
7	1.2722793	1.3159318	1.3608618	1.4071004	1.4546792	1.5036303	1.6057815	7
8	1.3168090	1.3685691	1.4221006	1.4774554	1.5346865	1.5938481	1.7181862	8
9	1.3628974	1.4233118	1.4860951	1.5513282	1.6190943	1.6894790	1.8384592	9
10	1.4105988	1.4802443	1.5529694	1.6288946	1.7081445	1.7908477	1.9671514	10
11	1.4599697	1.5394541	1.6228531	1.7103394	1.8020924	1.8982986	2.1048520	11
12	1.5110687	1.6010322	1.6958814	1.7958563	1.9012075	2.0121965	2.2521916	12
13	1.5639561	1.6650735	1.7721961	1.8856491	2.0057739	2.1329283	2.4098450	13
14	1.6186945	1.7316765	1.8519449	1.9799316	2.1160915	2.2609040	2.5785342	14
15	1.6753488	1.8009435	1.9352824	2.0789282	2.2324765	2.3965582	2.7590315	15
16	1.7339860	1.8729813	2.0223702	2.1828746	2.3552627	2.5403517	2.9521638	16
17	1.7946756	1.9473005	2.1133768	2.2920183	2.4848022	2.6927728	3.1588152	17
18	1.8574892	2.0258165	2.2084788	2.4066192	2.6214663	2.8543392	3.3799323	18
19	1.9225013	2.1068492	2.3078603	2.5269502	2.7656469	3.0255995	3.6165275	19
20	1.9897889	2.1911231	2.4117140	2.6532977	2.9177575	3.2071355	3.8696845	20
21	2.0594315	2.2787681	2.5202412	2.7859626	3.0782342	3.3995636	4.1405624	21
22	2.1315116	2.3699188	2.6336520	2.9252607	3.2475370	3.6035374	4.4304017	22
23	2.2061145	2.4647155	2.7521664	3.0715238	3.4261516	3.8197497	4.7405299	23
24	2.2833285	2.5633042	2.8760138	3.2250999	3.6145899	4.0489346	5.0723670	24
25	2.3632450	2.6658363	3.0054345	3.3863549	3.8133924	4.2918707	5.4274326	25
26	2.4459586	2.7724698	3.1406790	3.5556727	4.0231289	4.5493830	5.8073529	26
27	2.5315671	2.8833686	3.2820096	3.7334563	4.2444010	4.8223459	6.2138676	27
28	2.6201720	2.9987033	3.4297000	3.9201291	4.4778431	5.1116867	6.6488384	28
29	2.7118780	3.1186515	3.5840365	4.1161356	4.7241244	5.4183879	7.1142571	29
30	2.8067937	3.2433975	3.7453181	4.3219424	4.9839513	5.7434912	7.6122550	30
31	2.9050315	3.3731334	3.9138575	4.5380395	5.2580686	6.0881006	8.1451129	31
32	3.0067076	3.5080588	4.0899810	4.7649415	5.5472624	6.4533867	8.7152708	32
33	3.1119424	3.6483811	4.2740302	5.0031885	5.8523618	6.8405899	9.3253398	33
34	3.2208603	3.7943163	4.4663615	5.2533480	6.1742417	7.2510253	9.9781135	34
35	3.3335905	3.9460890	4.6673478	5.5160154	6.5138250	7.6860868	10.6765815	35
36	3.4502661	4.1039326	4.8773785	5.7918161	6.8720854	8.1472520	11.4239422	36
37	3.5710254	4.2680899	5.0968605	6.0814069	7.2500501	8.6360871	12.2236181	37
38	3.6960113	4.4388135	5.3262192	6.3854773	7.6488028	9.1542524	13.0792714	38
39	3.8253717	4.6163660	5.5658991	6.7047512	8.0694870	9.7035075	13.9948204	39
40	3.9592597	4.8010206	5.8163645	7.0399887	8.5133088	10.2857179	14.9744578	40
41	4.0978338	4.9930615	6.0781009	7.3919882	8.9815408	10.9028610	16.0226699	41
42	4.2412580	5.1927839	6.3516155	7.7615876	9.4755255	11.5570327	17.1442568	42
43	4.3897020	5.4004953	6.6374382	8.1496669	9.9966794	12.2504546	18.3443548	43
44	4.5433416	5.6165151	6.9361229	8.5571503	10.5464968	12.9854819	19.6284596	44
45	4.7023586	5.8411757	7.2482484	8.9850078	11.1265541	13.7646108	21.0024518	45
46	4.8669411	6.0748227	7.5744196	9.4342582	11.7385146	14.5904875	22.4726234	46
47	5.0372840	6.3178156	7.9152685	9.9059711	12.3841329	15.4659167	24.0457070	47
48	5.2135890	6.5705282	8.2714556	10.4012697	13.0652602	16.3938717	25.7289065	48
49	5.3960646	6.8333494	8.6436711	10.9213331	13.7838495	17.3775040	27.5299300	49
50	5.5849269	7.1066834	9.0326363	11.4673998	14.5419612	18.4201543	29.4570251	50
51	5.7803993	7.3909507	9.4391049	12.0407698	15.3417691	19.5253635	31.5190168	51
52	5.9827133	7.6865887	9.8638646	12.6428083	16.1855664	20.6968853	33.7253480	52
53	6.1921082	7.9940523	10.3077385	13.2749487	17.0757725	21.9386985	36.0861224	53
54	6.4088320	8.3138144	10.7715868	13.9386961	18.0149400	23.2550204	38.6121509	54
55	6.6331411	8.6463669	11.2563082	14.6356309	19.0057617	24.6503216	41.3150015	55
56	6.8653011	8.9922216	11.7628420	15.3674125	20.0510786	26.1293409	44.2070516	56
57	7.1055866	9.3519105	12.2921699	16.1357831	21.1538879	27.6971013	47.3015452	57
58	7.3542822	9.7259869	12.8453176	16.9425722	22.3173518	29.3589274	50.6126534	58
59	7.6116820	10.1150264	13.4233569	17.7897009	23.5448061	31.1204631	54.1555391	59
60	7.8780909	10.5196274	14.0274079	18.6791859	24.8397705	32.9876909	57.9464268	60

Table 14.—*The accumulation of an annuity of 1 per annum at the end of n years.*

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1
2	2.0150000	2.0175000	2.0200000	2.0225000	2.0250000	2.0275000	2.0300000	2
3	3.0452250	3.0528063	3.0604000	3.0680063	3.0756250	3.0832563	3.0909000	3
4	4.0909034	4.1062304	4.1216080	4.1370364	4.1525156	4.1680458	4.1836270	4
5	5.1522669	5.1780894	5.2040402	5.2301197	5.2563285	5.2826671	5.3091358	5
6	6.2295509	6.2687060	6.3081210	6.3477974	6.3877367	6.4279404	6.4684099	6
7	7.3229942	7.3784083	7.4342834	7.4906228	7.5474302	7.6047088	7.6624622	7
8	8.4328391	8.5075305	8.5829691	8.6591619	8.7361159	8.8138383	8.8923361	8
9	9.5593317	9.6564122	9.7546284	9.8539930	9.9545188	10.0562188	10.1591061	9
10	10.7027217	10.8253995	10.9497210	11.0757078	11.2033818	11.3327648	11.4638793	10
11	11.8632625	12.0148439	12.1687154	12.3249113	12.4834663	12.6444159	12.8077957	11
12	13.0412114	13.2251037	13.4120897	13.6022218	13.7955530	13.9921373	14.1920296	12
13	14.2368296	14.4565430	14.6803315	14.9082718	15.1404418	15.3769211	15.6177905	13
14	15.4503821	15.7095325	15.9739382	16.2437079	16.5189528	16.7997864	17.0863242	14
15	16.6821378	16.9844494	17.2934169	17.6091913	17.9319267	18.2617805	18.5989139	15
16	17.9323698	18.2816772	18.6392853	19.0053981	19.3802248	19.7639795	20.1568813	16
17	19.2013554	19.6016066	20.0120710	20.4330196	20.8647305	21.3074889	21.7615877	17
18	20.4893757	20.9446347	21.4123124	21.8927625	22.3863487	22.8934449	23.4144354	18
19	21.7967164	22.3111658	22.8405586	23.3853497	23.9460074	24.5230146	25.1168684	19
20	23.1236671	23.7016112	24.2973698	24.9115200	25.5446576	26.1973975	26.8703745	20
21	24.4705221	25.1163894	25.7833172	26.4720292	27.1832741	27.9178259	28.6764857	21
22	25.8375799	26.5559262	27.2989835	28.0676499	28.8628559	29.6855662	30.5367803	22
23	27.2251436	28.0206549	28.8449632	29.6991720	30.5844273	31.5019192	32.4528837	23
24	28.6335208	29.5110164	30.4218625	31.3674034	32.3490380	33.3682220	34.4264702	24
25	30.0630236	31.0274592	32.0302997	33.0731700	34.1577639	35.2858481	36.4592643	25
26	31.5139690	32.5704397	33.6709057	34.8173163	36.0117080	37.2562089	38.5530423	26
27	32.9866785	34.1404224	35.3443238	36.6007059	37.9120007	39.2807547	40.7096335	27
28	34.4814787	35.7378798	37.0512103	38.4242218	39.8598008	41.3609754	42.9309225	28
29	35.9987009	37.3632927	38.7922345	40.2887668	41.8562958	43.4984022	45.2188502	29
30	37.5386814	39.0171503	40.5680792	42.1952640	43.9027032	45.6946083	47.5754157	30
31	39.1017616	40.6999504	42.3794408	44.1446575	46.0002707	47.9512100	50.0026782	31
32	40.6882880	42.4121996	44.2270296	46.1379123	48.1502775	50.2698683	52.5027585	32
33	42.2986123	44.1544131	46.1115702	48.1760153	50.3540345	52.6522897	55.0778413	33
34	43.9330915	45.9271153	48.0338016	50.2599756	52.6128853	55.1002277	57.7301765	34
35	45.5920879	47.7308398	49.9944776	52.3908251	54.9282074	57.6154839	60.4620818	35
36	47.2759692	49.5661295	51.9943672	54.5696186	57.3014126	60.1999097	63.2759443	36
37	48.9851087	51.4335368	54.0342545	56.7974351	59.7339479	62.8554072	66.1742226	37
38	50.7198854	53.3336236	56.1149396	59.0753774	62.2272966	65.5839309	69.1594493	38
39	52.4806837	55.2669621	58.2372384	61.4045733	64.7829791	68.3874890	72.2342328	39
40	54.2678939	57.2341339	60.4019832	63.7861762	67.4025535	71.2681450	75.4012597	40
41	56.0819123	59.2357312	62.6100228	66.2213652	70.0876174	74.2280190	78.6632975	41
42	57.9231410	61.2723565	64.8622233	68.7113459	72.8398078	77.2692895	82.0231965	42
43	59.7919881	63.3446228	67.1594678	71.2573512	75.6608030	80.3941950	85.4838923	43
44	61.6888679	65.4531537	69.5026571	73.8606416	78.5523231	83.6050353	89.0484091	44
45	63.6142010	67.5985839	71.8927103	76.5225061	81.5161312	86.9041738	92.7198614	45
46	65.5684140	69.7815591	74.3305645	79.2442624	84.5540344	90.2940386	96.5014572	46
47	67.5519402	72.0027364	76.8171758	82.0272583	87.6678853	93.7771246	100.3965010	47
48	69.5652193	74.2627843	79.3535193	84.8728717	90.8595824	97.3559956	104.4083960	48
49	71.6086976	76.5623830	81.9405897	87.7825113	94.1310720	101.0332854	108.5406479	49
50	73.6828280	78.9022247	84.5794015	90.7576178	97.4843488	104.8117008	112.7968673	50
51	75.7880705	81.2830136	87.2709895	93.7996642	100.9214575	108.6940226	117.1807733	51
52	77.9248915	83.7054664	90.0164093	96.9101566	104.4444940	112.6831082	121.6961965	52
53	80.0937649	86.1703120	92.8167375	100.0906351	108.0556063	116.7818937	126.3470824	53
54	82.2951714	88.6782925	95.6730722	103.3426744	111.7569965	120.9933957	131.1374949	54
55	84.5295989	91.2301626	98.5865337	106.6678846	115.5509214	125.3207141	136.0716197	55
56	86.7975429	93.8266904	101.5582643	110.0679120	119.4396944	129.7670338	141.1537683	56
57	89.0995061	96.4686575	104.5894296	113.5444400	123.4256868	134.3356272	146.3883814	57
58	91.4359987	99.1568590	107.6812182	117.0991899	127.5113289	139.0298569	151.7800328	58
59	93.8075386	101.8921041	110.8348426	120.7339217	131.6991122	143.8531780	157.3334338	59
60	96.2146517	104.6752159	114.0515394	124.4504349	135.9915900	148.8091404	163.0534368	60

TABLE 14.—*The accumulation of an annuity of 1 per annum at the end of n years*—Con.

$$s_{\overline{n}|}=\frac{(1+i)^n-1}{i}.$$

Yrs.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Yrs.
1	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1
2	2.0350000	2.0400000	2.0450000	2.0500000	2.0550000	2.0600000	2.0700000	2
3	3.1062250	3.1216000	3.1370250	3.1525000	3.1680250	3.1836000	3.2149000	3
4	4.2149429	4.2464640	4.2781911	4.3101250	4.3422664	4.3746160	4.4399430	4
5	5.3624659	5.4163226	5.4707097	5.5256313	5.5810910	5.6370930	5.7507390	5
6	6.5501522	6.6329755	6.7168917	6.8019128	6.8880510	6.9753185	7.1532907	6
7	7.7794075	7.8982945	8.0191518	8.1420085	8.2668938	8.3938377	8.6540211	7
8	9.0516868	9.2142263	9.3800136	9.5491089	9.7215730	9.8974679	10.2598026	8
9	10.3684958	10.5827953	10.8021142	11.0265643	11.2562595	11.4913160	11.9779888	9
10	11.7313932	12.0061071	12.2882094	12.5778925	12.8753538	13.1807949	13.8164480	10
11	13.1419919	13.4863514	13.8411788	14.2067872	14.5834983	14.9716426	15.7835993	11
12	14.6019616	15.0258055	15.4640318	15.9171265	16.3855907	16.8699412	17.8884513	12
13	16.1130303	16.6268377	17.1599133	17.7129829	18.2867981	18.8821377	20.1406429	13
14	17.6769864	18.2919112	18.9321094	19.5986320	20.2925720	21.0150659	22.5504879	14
15	19.2956809	20.0235876	20.7840543	21.5785636	22.4086635	23.2759699	25.1290220	15
16	20.9710297	21.8245311	22.7193367	23.6574918	24.6411400	25.6725281	27.8880536	16
17	22.7050158	23.6975124	24.7417069	25.8403664	26.9964027	28.2128798	30.8402173	17
18	24.4996913	25.6454129	26.8550837	28.1323847	29.4812048	30.9056526	33.9990325	18
19	26.3571805	27.6712294	29.0635625	30.5390039	32.1026711	33.7599917	37.3789648	19
20	28.2796818	29.7780786	31.3714228	33.0659541	34.8683180	36.7855912	40.9954923	20
21	30.2694707	31.9692017	33.7831368	35.7192518	37.7860755	39.9927267	44.8651768	21
22	32.3289022	34.2479698	36.3033780	38.5052144	40.8643097	43.3922903	49.0057392	22
23	34.4604137	36.6178886	38.9370300	41.4304751	44.1118467	46.9958277	53.4361409	23
24	36.6665282	39.0826041	41.6891963	44.5019989	47.5379983	50.8155774	58.1766708	24
25	38.9498567	41.6459083	44.5652102	47.7270988	51.1525882	54.8645120	63.2490377	25
26	41.3131017	44.3117446	47.5706446	51.1134538	54.9659805	59.1563827	68.6764704	26
27	43.7590602	47.0842144	50.7113236	54.6691265	58.9891094	63.7057657	74.4838233	27
28	46.2906273	49.9675830	53.9933332	58.4025828	63.2335105	68.5281116	80.6976909	28
29	48.9107993	52.9662863	57.4230332	62.3227119	67.7113535	73.6397983	87.3465293	29
30	51.6226773	56.0849378	61.0070697	66.4388475	72.4354780	79.0581862	94.4607863	30
31	54.4294710	59.3283353	64.7523878	70.7607899	77.4194293	84.8016774	102.0730414	31
32	57.3345025	62.7014687	68.6662452	75.2988294	82.6774 79	90.8897780	110.2181543	32
33	60.3412101	66.2095274	72.7562263	80.0637708	88.2247603	97.3431647	118.9334251	33
34	63.4531524	69.8579085	77.0302565	85.0669594	94.0771221	104.1837546	128.2587648	34
35	66.6740127	73.6522249	81.4966180	90.3203074	100.2513638	111.4347799	138.2368784	35
36	70.0076032	77.5983139	86.1639658	95.8363227	106.7651888	119.1208667	148.9134598	36
37	73.4578693	81.7022464	91.0413443	101.6281389	113.6372742	127.2681187	160.3374020	37
38	77.0288947	85.9703363	96.1382048	107.7095458	120.8873243	135.9042058	172.5610202	38
39	80.7249060	90.4091497	101.4644240	114.0950231	128.5361271	145.0584581	185.6402916	39
40	84.5502778	95.0255157	107.0303231	120.7997742	136.6056141	154.7619656	199.6351120	40
41	88.5095375	99.8265363	112.8466876	127.8397630	145.1189229	165.0476836	214.6095698	41
42	92.6073713	104.8195978	118.9247885	135.2317511	154.1004636	175.9505446	230.6322397	42
43	96.8486293	110.0123817	125.2764040	142.9933387	163.5759891	187.5075772	247.7764695	43
44	101.2383313	115.4128770	131.9138422	151.1430056	173.5726685	199.7580319	266.1208513	44
45	105.7816729	121.0293920	138.8499651	159.7001559	184.1191653	212.7435138	285.7493108	45
46	110.4840315	126.8705677	146.0982135	168.6851637	195.2457194	226.5081246	306.7517626	46
47	115.3509726	132.9453904	153.6726331	178.1194219	206.9842339	241.0986121	329.2243860	47
48	120.3882566	139.2632060	161.5879016	188.0253929	219.3683668	256.5645288	353.2700930	48
49	125.6018456	145.8337343	169.8593572	198.4266626	232.4336270	272.9584006	378.9989995	49
50	130.9979102	152.6670837	178.5030283	209.3479957	246.2174765	290.3359046	406.5289295	50
51	136.5828370	159.7737670	187.5356646	220.8153955	260.7594377	308.7560589	435.9859545	51
52	142.3632363	167.1647177	196.9747695	232.8561653	276.1012067	328.2814224	467.5049714	52
53	148.3459496	174.8513064	206.8386341	245.4989735	292.2867731	348.9783077	501.2303194	53
54	154.5380578	182.8453587	217.1463726	258.7739222	309.3625456	370.9170062	537.3164417	54
55	160.9468898	191.1591730	227.9179594	272.7126183	327.3774856	394.1720266	575.9285926	55
56	167.5800310	199.8055399	239.1742676	287.3482492	346.3832473	418.8223482	617.2435941	56
57	174.4453321	208.7977615	250.9371096	302.7156617	366.4343259	444.9516891	661.4506457	57
58	181.5509187	218.1496720	263.2292795	318.8514448	387.5882139	472.6487904	708.7521909	58
59	188.9052009	227.8756589	276.0745971	335.7940170	409.9055656	502.0077178	759.3648443	59
60	196.5168829	237.9906852	289.4979540	353.5837179	433.4503717	533.1281809	813.5203834	60

TABLE 15.—*The annual sinking fund which will accumulate to 1 at the end of n years.*

$$\frac{1}{s_{\overline{n|}}}=\frac{i}{(1+i)^n-1}.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1
2	0.4962779	0.4956630	0.4950495	0.4944376	0.4938272	0.4932183	0.4926108	2
3	0.3283830	0.3275675	0.3267547	0.3259446	0.3251372	0.3243324	0.3235304	3
4	0.2444448	0.2435324	0.2426238	0.2417189	0.2408179	0.2399206	0.2390271	4
5	0.1940893	0.1931214	0.1921584	0.1912002	0.1902469	0.1892983	0.1883546	5
6	0.1605252	0.1595226	0.1585258	0.1575350	0.1565500	0.1555708	0.1545975	6
7	0.1365562	0.1355306	0.1345120	0.1335003	0.1324954	0.1314975	0.1305064	7
8	0.1185840	0.1175429	0.1165098	0.1154846	0.1144674	0.1134580	0.1124564	8
9	0.1046098	0.1035581	0.1025154	0.1014817	0.1004569	0.0994410	0.0984339	9
10	0.0934342	0.0923753	0.0913265	0.0902877	0.0892588	0.0882397	0.0872305	10
11	0.0842938	0.0832304	0.0821779	0.0811365	0.0801060	0.0790863	0.0780775	11
12	0.0766800	0.0756138	0.0745596	0.0735174	0.0724871	0.0714687	0.0704621	12
13	0.0702404	0.0691728	0.0681184	0.0670769	0.0660483	0.0650325	0.0640295	13
14	0.0647233	0.0636556	0.0626020	0.0615623	0.0605365	0.0595246	0.0585263	14
15	0.0599444	0.0588774	0.0578255	0.0567885	0.0557665	0.0547592	0.0537666	15
16	0.0557651	0.0546996	0.0536501	0.0526166	0.0515990	0.0505971	0.0496109	16
17	0.0520797	0.0510162	0.0499698	0.0489404	0.0479278	0.0469319	0.0459525	17
18	0.0488058	0.0477449	0.0467021	0.0456772	0.0446701	0.0436806	0.0427087	18
19	0.0458785	0.0448206	0.0437818	0.0427618	0.0417606	0.0407780	0.0398139	19
20	0.0432457	0.0421912	0.0411567	0.0401421	0.0391471	0.0381717	0.0372157	20
21	0.0408655	0.0398146	0.0387848	0.0377757	0.0367873	0.0358194	0.0348718	21
22	0.0387033	0.0376564	0.0366314	0.0356282	0.0346466	0.0336864	0.0327474	22
23	0.0367208	0.0356878	0.0346681	0.0336710	0.0326964	0.0317441	0.0308139	23
24	0.0349241	0.0338857	0.0328711	0.0318802	0.0309128	0.0299686	0.0290474	24
25	0.0332635	0.0322295	0.0312204	0.0302360	0.0292759	0.0283400	0.0274279	25
26	0.0317320	0.0307027	0.0296992	0.0287213	0.0277687	0.0268412	0.0259383	26
27	0.0303153	0.0292908	0.0282931	0.0273219	0.0263769	0.0254578	0.0245642	27
28	0.0290011	0.0279815	0.0269897	0.0260253	0.025[illegible]879	0.0241774	0.0232932	28
29	0.0277788	0.0267642	0.0257784	0.0248208	0.0238913	0.0229894	0.0221147	29
30	0.0266392	0.0256298	0.0246499	0.0236993	0.0227776	0.0218844	0.0210193	30
31	0.0255743	0.0245701	0.0235964	0.0226528	0.0217390	0.0208545	0.0199989	31
32	0.0245771	0.0235781	0.0226106	0.0216742	0.0207683	0.0198926	0.0190466	32
33	0.0236414	0.0226478	0.0216865	0.0207572	0.0198594	0.0189925	0.0181561	33
34	0.0227619	0.0217736	0.0208187	0.0198966	0.0190068	0.0181488	0.0173220	34
35	0.0219336	0.0209508	0.0200022	0.0190873	0.0182056	0.0173565	0.0165393	35
36	0.0211524	0.0201751	0.0192329	0.0183252	0.0174516	0.0166113	0.0158038	36
37	0.0204144	0.0194426	0.0185068	0.0176064	0.0167409	0.0159095	0.0151116	37
38	0.0197161	0.0187499	0.0178206	0.0169275	0.0160701	0.0152476	0.0144593	38
39	0.0190546	0.0180940	0.0171711	0.0162854	0.0154362	0.0146226	0.0138439	39
40	0.0184271	0.0174721	0.0165558	0.0156774	0.0148362	0.0140315	0.0132624	40
41	0.0178311	0.0168817	0.0159719	0.0151009	0.0142679	0.0134720	0.0127124	41
42	0.0172643	0.0163206	0.0154173	0.0145536	0.0137288	0.0129418	0.0121917	42
43	0.0167247	0.0157867	0.0148899	0.0140336	0.0132169	0.0124387	0.0116981	43
44	0.0162104	0.0152781	0.0143879	0.0135390	0.0127304	0.0119610	0.0112298	44
45	0.0157198	0.0147932	0.0139096	0.0130681	0.0122675	0.0115069	0.0107852	45
46	0.0152512	0.0143304	0.0134534	0.0126192	0.0118268	0.0110749	0.0103625	46
47	0.0148034	0.0138884	0.0130179	0.0121911	0.0114067	0.0106656	0.0099605	47
48	0.0143750	0.0134657	0.0126018	0.0117823	0.0110060	0.0102716	0.0095778	48
49	0.0139648	0.0130612	0.0122040	0.0113918	0.0106235	0.0098977	0.0092131	49
50	0.0135717	0.0126739	0.0118232	0.0110184	0.0102581	0.0095409	0.0088655	50
51	0.0131947	0.0123027	0.0114586	0.0106610	0.0099087	0.0092001	0.0085338	51
52	0.0128329	0.0119467	0.0111091	0.0103188	0.0095745	0.0088744	0.0082172	52
53	0.0124854	0.0116049	0.0107739	0.0099909	0.0092545	0.0085630	0.0079147	53
54	0.0121514	0.0112767	0.0104523	0.0096765	0.0089480	0.0082649	0.0076256	54
55	0.0118302	0.0109613	0.0101434	0.0093749	0.0086542	0.0079795	0.0073491	55
56	0.0115211	0.0106580	0.0098466	0.0090853	0.0083724	0.0077061	0.0070845	56
57	0.0112234	0.0103661	0.0095612	0.0088071	0.0081020	0.0074440	0.0068311	57
58	0.0109366	0.0100850	0.0092867	0.0085398	0.0078424	0.0071927	0.0065885	58
59	0.0106601	0.0098143	0.0090224	0.0082827	0.0075931	0.0069515	0.0063559	59
60	0.0103934	0.0095534	0.0087680	0.0080353	0.0073534	0.0067200	0.0061330	60

TABLE 15.—*The annual sinking fund which will accumulate to 1 at the end of n years*—Continued.

$$\frac{1}{s_{\overline{n}|}}=\frac{i}{(1+i)^n-1}.$$

Years.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Years.
1	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1
2	0.4914005	0.4901961	0.4889976	0.4878049	0.4866180	0.4854369	0.4830918	2
3	0.3219342	0.3203485	0.3187734	0.3172086	0.3156541	0.3141098	0.3110517	3
4	0.2372511	0.2354901	0.2337437	0.2320118	0.2302945	0.2285915	0.2252281	4
5	0.1864814	0.1846271	0.1827916	0.1809748	0.1791764	0.1773964	0.1738907	5
6	0.1526682	0.1507619	0.1488784	0.1470175	0.1451790	0.1433626	0.1397958	6
7	0.1285445	0.1266096	0.1247015	0.1228198	0.1209644	0.1191350	0.1155532	7
8	0.1104767	0.1085278	0.1066097	0.1047218	0.1028640	0.1010359	0.0974678	8
9	0.0964460	0.0944930	0.0925745	0.0906901	0.0888395	0.0870222	0.0834865	9
10	0.0852414	0.0832909	0.0813788	0.0795046	0.0776678	0.0758680	0.0723775	10
11	0.0760920	0.0741490	0.0722482	0.0703889	0.0685707	0.0667929	0.0633569	11
12	0.0684840	0.0665522	0.0646662	0.0628254	0.0610292	0.0592770	0.0559020	12
13	0.0620616	0.0601437	0.0582754	0.0564558	0.0546843	0.0529601	0.0496509	13
14	0.0565707	0.0546690	0.0528203	0.0510240	0.0492791	0.0475849	0.0443449	14
15	0.0518251	0.0499411	0.0481138	C.0463423	0.0446256	0.0429628	0.0397946	15
16	0.0476848	0.0458200	0.0440154	0.0422699	0.0405825	0.0389521	0.0358577	16
17	0.0440431	0.0421985	0.0404176	0.0386991	0.0370420	0.0354448	0.0324252	17
18	0.0408168	0.0389933	0.0372369	0.0355462	0.0339199	0.0323565	0.0294126	18
19	0.0379403	0.0361386	0.0344073	0.0327450	0.0311501	0.0296209	0.0267530	19
20	0.0353611	0.0335818	0.0318761	0.0302426	0.0286793	0.0271846	0.0243929	20
21	0.0330366	0.0312801	0.0296006	0.0279961	0.0264648	0.0250046	0.0222890	21
22	0.0309321	0.0291988	0.0275457	0.0259705	0.0244712	0.0230456	0.0204058	22
23	0.0290188	0.0273091	0.0256825	0.0241368	0.0226696	0.0212785	0.0187139	23
24	0.0272728	0.0255868	0.0239870	0.0224709	0.0210358	0.0196790	0.0171890	24
25	0.0256740	0.0240120	0.0224390	0.0209525	0.0195494	0.0182267	0.0158105	25
26	0.0242054	0.0225674	0.0210214	0.0195643	0.0181931	0.0169044	0.0145610	26
27	0.0228524	0.0212385	0.0197195	0.0182919	0.0169523	0.0156972	0.0134257	27
28	0.0216027	0.0200130	0.0185208	0.0171225	0.0158144	0.0145926	0.0123919	28
29	0.0204454	0.0188799	0.0174146	0.0160455	0.0147686	0.0135796	0.0114487	29
30	0.0193713	0.0178301	0.0163915	0.0150514	0.0138054	0.0126489	0.0105864	30
31	0.0183724	0.0168554	0.0154435	0.0141321	0.0129167	0.0117922	0.0097969	31
32	0.0174415	0.0159486	0.0145632	0.0132804	0.0120952	0.0110023	0.0090729	32
33	0.0165724	0.0151036	0.0137445	0.0124900	0.0113347	0.0102729	0.0084081	33
34	0.0157597	0.0143148	0.0129819	0.0117554	0.0106296	0.0095984	0.0077967	34
35	0.0149984	0.0135773	0.0122705	0.0110717	0.0099749	0.0089739	0.0072340	35
36	0.0142842	0.0128869	0.0116058	0.0104345	0.0093664	0.0083948	0.0067153	36
37	0.0136133	0.0122396	0.0109840	0.0098398	0.0087999	0.0078574	0.0062369	37
38	0.0129821	0.0116319	0.0104017	0.0092842	0.0082722	0.0073581	0.0057951	38
39	0.0123878	0.0110608	0.0098557	0.0087646	0.0077799	0.0068938	0.0053868	39
40	0.0118273	0.0105235	0.0093432	0.0082782	0.0073203	0.0064615	0.0050091	40
41	0.0112982	0.0100174	0.0088616	0.0078223	0.0068909	0.0060589	0.0046596	41
42	0.0107983	0.0095402	0.0084087	0.0073947	0.0064893	0.0056834	0.0043359	42
43	0.0103254	0.0090899	0.0079824	0.0069933	0.0061134	0.0053331	0.0040359	43
44	0.0098777	0.0086645	0.0075807	0.0066163	0.0057613	0.0050061	0.0037577	44
45	0.0094534	0.0082625	0.0072020	0.0062617	0.0054313	0.0047005	0.0034996	45
46	0.0090511	0.0078821	0.0068447	0.0059282	0.0051218	0.0044149	0.0032600	46
47	0.0086692	0.0075219	0.0065073	0.0056142	0.0048313	0.0041477	0.0030374	47
48	0.0083065	0.0071807	0.0061886	0.0053184	0.0045585	0.0038977	0.0028307	48
49	0.0079617	0.0068571	0.0058872	0.0050397	0.0043023	0.0036636	0.0026385	49
50	0.0076337	0.0065502	0.0056022	0.0047767	0.0040615	0.0034443	0.0024599	50
51	0.0073216	0.0062589	0.0053323	0.0045287	0.0038350	0.0032388	0.0022937	51
52	0.0070243	0.0059821	0.0050768	0.0042945	0.0036219	0.0030462	0.0021390	52
53	0.0067410	0.0057192	0.0048347	0.0040733	0.0034213	0.0028655	0.0019951	53
54	0.0064709	0.0054691	0.0046052	0.0038644	0.0032325	0.0026960	0.0018611	54
55	0.0062132	0.0052312	0.0043875	0.0036669	0.0030546	0.0025370	0.0017363	55
56	0.0059673	0.0050049	0.0041811	0.0034801	0.0028870	0.0023877	0.0016201	56
57	0.0057325	0.0047893	0.0039851	0.0033034	0.0027290	0.0022474	0.0015118	57
58	0.0055081	0.0045840	0.0037990	0.0031363	0.0025801	0.0021157	0.0014109	58
59	0.0052937	0.0043884	0.0036222	0.0029780	0.0024396	0.0019920	0.0013169	59
60	0.0050886	0.0042019	0.0034543	0.0028282	0.0023071	0.0018757	0.0012292	60

TABLE 16.—*The present value of 1 due in n years.*

$$v^n = (1+i)^{-n}.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	0.9852217	0.9828010	0.9803922	0.9779951	0.9756098	0.9732360	0.9708738	1
2	0.9706618	0.9658978	0.9611688	0.9564744	0.9518144	0.9471883	0.9425959	2
3	0.9563170	0.9492853	0.9423223	0.9354273	0.9285994	0.9218378	0.9151417	3
4	0.9421842	0.9329585	0.9238454	0.9148434	0.9059506	0.8971657	0.8884871	4
5	0.9282603	0.9169125	0.9057308	0.8947123	0.8838543	0.8731540	0.8626088	5
6	0.9145422	0.9011425	0.8879714	0.8750243	0.8622969	0.8497849	0.8374843	6
7	0.9010268	0.8856438	0.8705602	0.8557695	0.8412652	0.8270413	0.8130915	7
8	0.8877111	0.8704116	0.8534904	0.8369384	0.8207466	0.8049064	0.7894092	8
9	0.8745922	0.8554414	0.8367553	0.8185216	0.8007284	0.7833639	0.7664167	9
10	0.8616672	0.8407286	0.8203483	0.8005101	0.7811984	0.7623979	0.7440939	10
11	0.8489332	0.8262689	0.8042630	0.7828950	0.7621448	0.7419931	0.7224213	11
12	0.8363874	0.8120579	0.7884932	0.7656675	0.7435559	0.7221344	0.7013799	12
13	0.8240270	0.7980913	0.7730325	0.7488191	0.7254204	0.7028072	0.6809513	13
14	0.8118493	0.7843649	0.7578750	0.7323414	0.7077272	0.6839973	0.6611178	14
15	0.7998515	0.7708746	0.7430147	0.7162263	0.6904656	0.6656908	0.6418620	15
16	0.7880310	0.7576163	0.7284458	0.7004658	0.6736249	0.6478742	0.6231669	16
17	0.7763853	0.7445861	0.7141626	0.6850521	0.6571951	0.6305345	0.6050165	17
18	0.7649116	0.7317799	0.7001594	0.6699776	0.6411659	0.6136589	0.5873946	18
19	0.7536075	0.7191940	0.6864308	0.6552348	0.6255277	0.5972350	0.5702860	19
20	0.7424704	0.7068246	0.6729713	0.6408165	0.6102709	0.5812506	0.5536758	20
21	0.7314980	0.6946679	0.6597758	0.6267154	0.5953863	0.5656940	0.5375493	21
22	0.7206876	0.6827203	0.6468390	0.6129246	0.5808647	0.5505538	0.5218925	22
23	0.7100371	0.6709782	0.6341559	0.5994372	0.5666972	0.5358187	0.5066918	23
24	0.6995439	0.6594380	0.6217215	0.5862467	0.5528754	0.5214781	0.4919337	24
25	0.6892058	0.6480963	0.6095309	0.5733464	0.5393906	0.5075213	0.4776056	25
26	0.6790205	0.6369497	0.5975793	0.5607300	0.5262347	0.4939380	0.4636947	26
27	0.6689857	0.6259948	0.5858620	0.5483912	0.5133997	0.4807182	0.4501891	27
28	0.6590993	0.6152283	0.5743746	0.5363239	0.5008778	0.4678523	0.4370768	28
29	0.6493589	0.6046470	0.5631123	0.5245221	0.4886613	0.4553307	0.4243464	29
30	0.6397624	0.5942476	0.5520709	0.5129801	0.4767427	0.4431442	0.4119868	30
31	0.6303078	0.5840272	0.5412460	0.5016920	0.4651148	0.4312839	0.3999872	31
32	0.6209929	0.5739825	0.5306333	0.4906523	0.4537706	0.4197410	0.3883370	32
33	0.6118157	0.5641105	0.5202287	0.4798556	0.4427030	0.4085071	0.3770263	33
34	0.6027741	0.5544084	0.5100282	0.4692964	0.4319053	0.3975738	0.3660449	34
35	0.5938661	0.5448731	0.5000276	0.4589696	0.4213711	0.3869331	0.3553834	35
36	0.5850897	0.5355018	0.4902232	0.4488700	0.4110937	0.3765773	0.3450324	36
37	0.5764431	0.5262917	0.4806109	0.4389927	0.4010671	0.3664986	0.3349829	37
38	0.5679242	0.5172400	0.4711872	0.4293327	0.3912849	0.3566896	0.3252262	38
39	0.5595313	0.5083440	0.4619482	0.4198853	0.3817414	0.3471432	0.3157536	39
40	0.5512623	0.4996010	0.4528904	0.4106458	0.3724306	0.3378522	0.3065568	40
41	0.5431156	0.4910083	0.4440102	0.4016095	0.3633470	0.3288100	0.2976280	41
42	0.5350893	0.4825635	0.4353041	0.3927722	0.3544848	0.3200097	0.2889592	42
43	0.5271815	0.4742639	0.4267688	0.3841293	0.3458389	0.3114450	0.2805429	43
44	0.5193907	0.4661070	0.4184007	0.3756765	0.3374038	0.3031094	0.2723718	44
45	0.5117149	0.4580904	0.4101968	0.3674098	0.3291744	0.2949970	0.2644386	45
46	0.5041527	0.4502117	0.4021537	0.3593250	0.3211458	0.2871017	0.2567365	46
47	0.4967021	0.4424685	0.3942684	0.3514181	0.3133129	0.2794177	0.2492588	47
48	0.4893617	0.4348585	0.3865376	0.3436852	0.3056712	0.2719394	0.2419988	48
49	0.4821298	0.4273793	0.3789584	0.3361224	0.2982158	0.2646612	0.2349503	49
50	0.4750047	0.4200288	0.3715279	0.3287261	0.2909422	0.2575778	0.2281071	50
51	0.4679849	0.4128048	0.3642430	0.3214925	0.2838461	0.2506840	0.2214632	51
52	0.4610689	0.4057049	0.3571010	0.3144181	0.2769230	0.2439747	0.2150128	52
53	0.4542551	0.3987272	0.3500990	0.3074994	0.2701688	0.2374450	0.2087503	53
54	0.4475419	0.3918695	0.3432343	0.3007329	0.2635793	0.2310900	0.2026702	54
55	0.4409280	0.3851297	0.3365043	0.2941153	0.2571505	0.2249051	0.1967672	55
56	0.4344118	0.3785059	0.3299061	0.2876433	0.2508786	0.2188858	0.1910361	56
57	0.4279919	0.3719959	0.3234374	0.2813137	0.2447596	0.2130275	0.1854719	57
58	0.4216669	0.3655980	0.3170955	0.2751235	0.2387898	0.2073260	0.1800698	58
59	0.4154354	0.3593100	0.3108779	0.2690694	0.2329657	0.2017772	0.1748251	59
60	0.4092960	0.3531303	0.3047823	0.2631486	0.2272836	0.1963768	0.1697331	60

TABLE 16.—*The present value of 1 due in n years*—Continued.

$$v^n = (1+i)^{-n}.$$

Years.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Years.
1	0.9661836	0.9615385	0.9569378	0.9523810	0.9478673	0.9433962	0.9345794	1
2	0.9335107	0.9245562	0.9157300	0.9070295	0.8984524	0.8899964	0.8734387	2
3	0.9019427	0.8889964	0.8762966	0.8638376	0.8516137	0.8396193	0.8162979	3
4	0.8714422	0.8548042	0.8385613	0.8227025	0.8072167	0.7920937	0.7628952	4
5	0.8419732	0.8219271	0.8024511	0.7835262	0.7651344	0.7472582	0.7129862	5
6	0.8135006	0.7903145	0.7678957	0.7462154	0.7252458	0.7049605	0.6663422	6
7	0.7859910	0.7599178	0.7348285	0.7106813	0.6874368	0.6650571	0.6227497	7
8	0.7594116	0.7306902	0.7031851	0.6768394	0.6515989	0.6274124	0.5820091	8
9	0.7337310	0.7025867	0.6729044	0.6446089	0.6176293	0.5918985	0.5439337	9
10	0.7089188	0.6755642	0.6439277	0.6139133	0.5854306	0.5583948	0.5083493	10
11	0.6849457	0.6495809	0.6161987	0.5846793	0.5549105	0.5267875	0.4750928	11
12	0.6617833	0.6245971	0.5896639	0.5568374	0.5259815	0.4969694	0.4440120	12
13	0.6394042	0.6005741	0.5642716	0.5303214	0.4985607	0.4688390	0.4149645	13
14	0.6177818	0.5774751	0.5399729	0.5050680	0.4725694	0.4423010	0.3878172	14
15	0.5968906	0.5552645	0.5167204	0.4810171	0.4479331	0.4172651	0.3624460	15
16	0.5767059	0.5339082	0.4944693	0.4581115	0.4245811	0.3936463	0.3387346	16
17	0.5572038	0.5133733	0.4731764	0.4362967	0.4024465	0.3713644	0.3165744	17
18	0.5383611	0.4936281	0.4528004	0.4155207	0.3814659	0.3503438	0.2958639	18
19	0.5201557	0.4746424	0.4333018	0.3957340	0.3615791	0.3305130	0.2765083	19
20	0.5025659	0.4563870	0.4146429	0.3768895	0.3427290	0.3118047	0.2584190	20
21	0.4855709	0.4388336	0.3967874	0.3589424	0.3248616	0.2941554	0.2415131	21
22	0.4691506	0.4219554	0.3797009	0.3418499	0.3079257	0.2775051	0.2257132	22
23	0.4532856	0.4057263	0.3633501	0.3255713	0.2918727	0.2617973	0.2109469	23
24	0.4379571	0.3901215	0.3477035	0.3100679	0.2766566	0.2469786	0.1971466	24
25	0.4231470	0.3751168	0.3327306	0.2953028	0.2622337	0.2329986	0.1842492	25
26	0.4088377	0.3606892	0.3184025	0.2812407	0.2485628	0.2198100	0.1721955	26
27	0.3950122	0.3468166	0.3046914	0.2678483	0.2356045	0.2073680	0.1609304	27
28	0.3816543	0.3334775	0.2915707	0.2550936	0.2233218	0.1956301	0.1504022	28
29	0.3687482	0.3206514	0.2790150	0.2429463	0.2116794	0.1845567	0.1405628	29
30	0.3562784	0.3083187	0.2670000	0.2313775	0.2006440	0.1741101	0.1313671	30
31	0.3442304	0.2964603	0.2555024	0.2203595	0.1901839	0.1642548	0.1227730	31
32	0.3325897	0.2850579	0.2444999	0.2098662	0.1802691	0.1549574	0.1147411	32
33	0.3213427	0.2740942	0.2339712	0.1998725	0.1708712	0.1461862	0.1072347	33
34	0.3104761	0.2635521	0.2238959	0.1903548	0.1619632	0.1379115	0.1002193	34
35	0.2999769	0.2534155	0.2142544	0.1812903	0.1535196	0.1301052	0.0936629	35
36	0.2898327	0.2436687	0.2050282	0.1726574	0.1455162	0.1227408	0.0875355	36
37	0.2800316	0.2342969	0.1961992	0.1644356	0.1379301	0.1157932	0.0818088	37
38	0.2705619	0.2252854	0.1877504	0.1566054	0.1307394	0.1092389	0.0764569	38
39	0.2614125	0.2166206	0.1796655	0.1491480	0.1239236	0.1030555	0.0714550	39
40	0.2525725	0.2082890	0.1719287	0.1420457	0.1174631	0.0972222	0.0667804	40
41	0.2440314	0.2002779	0.1645251	0.1352816	0.1113395	0.0917191	0.0624116	41
42	0.2357791	0.1925749	0.1574403	0.1288396	0.1055350	0.0865274	0.0583286	42
43	0.2278059	0.1851682	0.1506605	0.1227044	0.1000332	0.0816296	0.0545127	43
44	0.2201023	0.1780464	0.1441728	0.1168613	0.0948182	0.0770091	0.0509464	44
45	0.2126592	0.1711984	0.1379644	0.1112965	0.0898751	0.0726501	0.0476135	45
46	0.2054679	0.1646139	0.1320233	0.1059967	0.0851897	0.0685378	0.0444986	46
47	0.1985197	0.1582826	0.1263381	0.1009492	0.0807485	0.0646583	0.0415875	47
48	0.1918065	0.1521948	0.1208977	0.0961421	0.0765389	0.0609984	0.0388668	48
49	0.1853202	0.1463411	0.1156916	0.0915639	0.0725487	0.0575457	0.0363241	49
50	0.1790534	0.1407126	0.1107097	0.0872037	0.0687665	0.0542884	0.0339478	50
51	0.1729984	0.1353006	0.1059423	0.0830512	0.0651815	0.0512154	0.0317269	51
52	0.1671482	0.1300967	0.1013801	0.0790964	0.0617834	0.0483165	0.0296513	52
53	0.1614959	0.1250930	0.0970145	0.0753299	0.0585625	0.0455816	0.0277115	53
54	0.1560347	0.1202817	0.0928368	0.0717427	0.0555095	0.0430015	0.0258986	54
55	0.1507581	0.1156555	0.0888391	0.0683264	0.0526156	0.0405674	0.0242043	55
56	0.1456600	0.1112072	0.0850135	0.0650728	0.0498726	0.0382712	0.0226208	56
57	0.1407343	0.1069300	0.0813526	0.0619741	0.0472726	0.0361049	0.0211410	57
58	0.1359752	0.1028173	0.0778494	0.0590229	0.0448082	0.0340612	0.0197579	58
59	0.1313770	0.0988628	0.0744970	0.0562123	0.0424722	0.0321332	0.0184653	59
60	0.1269343	0.0950604	0.0712890	0.0535355	0.0402580	0.0303143	0.0172573	60

TABLE 17.—*The present value of an annuity of 1 for n years.*

$$a_{\overline{n}|} = \frac{1-v^n}{i}.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	0.9852217	0.9828010	0.9803922	0.9779951	0.9756098	0.9732360	0.9708738	1
2	1.9558834	1.9486988	1.9415609	1.9344696	1.9274242	1.9204243	1.9134697	2
3	2.9122004	2.8979840	2.8838833	2.8698969	2.8560236	2.8422621	2.8286114	3
4	3.8543847	3.8309425	3.8077287	3.7847402	3.7619742	3.7394279	3.7170984	4
5	4.7826450	4.7478551	4.7134595	4.6794525	4.6458285	4.6125819	4.5797072	5
6	5.6971872	5.6489976	5.6014309	5.5544768	5.5081254	5.4623668	5.4171914	6
7	6.5982140	6.5346414	6.4719911	6.4102463	6.3493906	6.2894081	6.2302830	7
8	7.4859251	7.4050530	7.3254814	7.2471846	7.1701372	7.0943144	7.0196922	8
9	8.3605173	8.2604943	8.1622367	8.0657062	7.9708655	7.8776783	7.7861089	9
10	9.2221846	9.1012229	8.9825850	8.8662164	8.7520639	8.6400762	8.5302028	10
11	10.0711178	9.9274918	9.7868481	9.6491113	9.5142087	9.3820693	9.2526241	11
12	10.9075052	10.7395497	10.5753412	10.4147788	10.2577646	10.1042037	9.9540040	12
13	11.7315322	11.5376410	11.3483738	11.1635979	10.9831850	10.8070109	10.6349553	13
14	12.5433815	12.3220059	12.1062488	11.8959392	11.6909122	11.4910081	11.2960731	14
15	13.3432330	13.0928805	12.8492635	12.6121655	12.3813777	12.1566989	11.9379351	15
16	14.1312641	13.8504968	13.5777093	13.3126313	13.0550027	12.8045732	12.5611020	16
17	14.9076493	14.5950828	14.2918719	13.9976834	13.7121977	13.4351077	13.1661185	17
18	15.6725609	15.3268627	14.9920313	14.6676611	14.3533636	14.0487666	13.7535131	18
19	16.4261684	16.0460567	15.6784620	15.3228959	14.9788913	14.6460016	14.3237991	19
20	17.1686388	16.7528813	16.3514333	15.9637124	15.5891623	15.2272521	14.8774749	20
21	17.9001367	17.4475492	17.0112092	16.5904278	16.1845486	15.7929461	15.4150241	21
22	18.6208244	18.1302695	17.6580482	17.2033523	16.7654132	16.3434999	15.9369166	22
23	19.3308615	18.8012476	18.2922041	17.8027896	17.3321105	16.8793186	16.4436084	23
24	20.0304054	19.4606857	18.9139256	18.3890362	17.8849858	17.4007967	16.9355421	24
25	20.7196112	20.1087820	19.5234565	18.9623826	18.4243764	17.9083180	17.4131477	25
26	21.3986317	20.7457317	20.1210358	19.5231126	18.9506111	18.4022559	17.8768424	26
27	22.0676175	21.3717264	20.7068978	20.0715038	19.4640109	18.8829741	18.3270315	27
28	22.7267167	21.9869547	21.2812724	20.6078276	19.9648887	19.3508264	18.7641082	28
29	23.3760756	22.5916017	21.8443847	21.1323498	20.4535499	19.8061571	19.1884546	29
30	24.0158380	23.1858493	22.3964556	21.6453299	20.9302926	20.2493013	19.6004414	30
31	24.6461458	23.7698765	22.9377015	22.1470219	21.3954074	20.6805852	20.0004285	31
32	25.2671387	24.3438590	23.4683348	22.6376742	21.8491780	21.1003262	20.3887655	32
33	25.8789544	24.9079695	23.9885636	23.1175298	22.2918809	21.5088333	20.7657918	33
34	26.4817285	25.4623779	24.4985917	23.5868262	22.7237863	21.9064071	21.1318367	34
35	27.0755946	26.0072510	24.9986193	24.0457958	23.1451573	22.2933403	21.4872201	35
36	27.6606843	26.5427528	25.4888425	24.4946658	23.5562511	22.6699175	21.8322525	36
37	28.2371274	27.0690446	25.9694534	24.9336585	23.9573181	23.0364161	22.1672354	37
38	28.8050516	27.5862846	26.4406406	25.3629912	24.3486030	23.3931057	22.4924616	38
39	29.3645829	28.0946286	26.9025888	25.7828765	24.7303444	23.7402488	22.8082151	39
40	29.9158452	28.5942296	27.3554792	26.1935222	25.1027751	24.0781011	23.1147720	40
41	30.4589608	29.0852379	27.7994895	26.5951317	25.4661220	24.4069110	23.4124000	41
42	30.9940500	29.5678014	28.2347936	26.9879039	25.8206068	24.7269207	23.7013592	42
43	31.5212316	30.0420652	28.6615623	27.3720332	26.1664457	25.0383656	23.9819021	43
44	32.0406222	30.5081722	29.0799631	27.7477097	26.5038495	25.3414751	24.2542739	44
45	32.5523372	30.9662626	29.4901599	28.1151195	26.8330239	25.6364721	24.5187125	45
46	33.0564898	31.4164743	29.8923136	28.4744445	27.1541696	25.9235738	24.7754491	46
47	33.5531920	31.8589428	30.2865820	28.8258626	27.4674826	26.2029915	25.0247078	47
48	34.0425537	32.2938013	30.6731196	29.1695478	27.7731537	26.4749309	25.2667066	48
49	34.5246834	32.7211806	31.0520780	29.5056702	28.0713695	26.7395922	25.5016569	49
50	34.9996881	33.1412095	31.4236059	29.8343963	28.3623117	26.9971700	25.7297640	50
51	35.4676730	33.5540142	31.7878489	30.1558888	28.6461577	27.2478540	25.9512272	51
52	35.9287419	33.9597191	32.1449499	30.4703069	28.9230807	27.4918287	26.1662400	52
53	36.3829969	34.3584463	32.4950489	30.7778062	29.1932495	27.7292737	26.3749903	53
54	36.8305388	34.7503158	32.8382833	31.0785391	29.4568288	27.9603637	26.5776605	54
55	37.2714668	35.1354455	33.1747875	31.3726544	29.7139793	28.1852688	26.7744276	55
56	37.7058786	35.5139514	33.5046937	31.6602977	29.9648578	28.4041545	26.9654637	56
57	38.1338706	35.8859473	33.8281310	31.9416114	30.2096174	28.6171820	27.1509357	57
58	38.5555375	36.2515452	34.1452265	32.2167349	30.4484072	28.8245081	27.3310055	58
59	38.9709729	36.6108553	34.4561044	32.4858043	30.6813729	29.0262852	27.5058306	59
60	39.3802689	36.9639855	34.7608867	32.7489529	30.9086565	29.2226620	27.6755637	60

Table 17.—*The present value of an annuity of 1 for n years*—Continued.

$$a_{\overline{n}|}=\frac{1-v^n}{i}.$$

Years.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Years.
1	0.9661836	0.9615385	0.9569378	0.9523810	0.9478673	0.9433962	0.9345794	1
2	1.8996943	1.8860947	1.8726678	1.8594104	1.8463197	1.8333927	1.8080182	2
3	2.8016370	2.7750910	2.7489644	2.7232480	2.6979334	2.6730120	2.6243160	3
4	3.6730792	3.6298952	3.5875257	3.5459505	3.5051501	3.4651056	3.3872113	4
5	4.5150524	4.4518223	4.3899767	4.3294767	4.2702845	4.2123638	4.1001974	5
6	5.3285530	5.2421369	5.1578725	5.07569 3	4.9955303	4.9173243	4.7665397	6
7	6.1145440	6.0020547	5.8927009	5.7863731	5.6829671	5.5823814	5.3892894	7
8	6.8739555	6.7327449	6.5958861	6.4632128	6.3345660	6.2097938	5.9712985	8
9	7.6076865	7.4353316	7.2687905	7.1078217	6.9521953	6.8016923	6.5152323	9
10	8.3166053	8.1108958	7.9127182	7.7217349	7.5376258	7.3600871	7.0235816	10
11	9.0015510	8.7604767	8.5289169	8.3064142	8.0925363	7.8868746	7.4986744	11
12	9.6633343	9.3850738	9.1185808	8.8632516	8.6185179	8.3838439	7.9426863	12
13	10.3027385	9.9856479	9.6828524	9.3935730	9.1170785	8.8526830	8.3576508	13
14	10.9205203	10.5631229	10.2228253	9.8986409	9.5896479	9.2949839	8.7454680	14
15	11.5174109	11.1183874	10.7395457	10.3796580	10.0375809	9.7122490	9.1079140	15
16	12.0941168	11.6522956	11.2340151	10.8377696	10.4621620	10.1058953	9.4466486	16
17	12.6513206	12.1656689	11.7071914	11.2740663	10.8646086	10.4772597	9.7632230	17
18	13.1896817	12.6592970	12.1599918	11.6895869	11.2460745	10.8276035	10.0590869	18
19	13.7098374	13.1339394	12.5932936	12.0853209	11.6076535	11.1581165	10.3355953	19
20	14.2124033	13.5903263	13.0079365	12.4622103	11.9503825	11.4699212	10.5940143	20
21	14.6979742	14.0291600	13.4047239	12.8211527	12.2752441	11.7640766	10.8355273	21
22	15.1671248	14.4511153	13.7844248	13.1630026	12.5831697	12.0415817	11.0612405	22
23	15.6204105	14.8568417	14.1477749	13.4885739	12.8750424	12.3033790	11.2721874	23
24	16.0583676	15.2469631	14.4954784	13.7986418	13.1516990	12.5503575	11.4693340	24
25	16.4815146	15.6220799	14.8282090	14.0939446	13.4139327	12.7833562	11.6535832	25
26	16.8903523	15.9827692	15.1466115	14.3751853	13.6624954	13.0031662	11.8257787	26
27	17.2853645	16.3295858	15.4513028	14.6430336	13.8980999	13.2105341	11.9867091	27
28	17.6670189	16.6630632	15.7428735	14.8981273	14.1214217	13.4061643	12.1371113	28
29	18.0357670	16.9837146	16.0218885	15.1410736	14.3331012	13.5907210	12.2776741	29
30	18.3920454	17.2920333	16.2888885	15.3724510	14.5337452	13.7648312	12.4090412	30
31	18.7362758	17.5884936	16.5443910	15.5928105	14.7239291	13.9290860	12.5318142	31
32	19.0688655	17.8735515	16.7888909	15.8026767	14.9041982	14.0840434	12.6465553	32
33	19.3902082	18.1476457	17.0228621	16.0025492	15.0750694	14.2302296	12.7537900	33
34	19.7006842	18.4111978	17.2467580	16.1929040	15.2370326	14.3681411	12.8540094	34
35	20.0006611	18.6646132	17.4610124	16.3741943	15.3905522	14.4982464	12.9476723	35
36	20.2904938	18.9082820	17.6660406	16.5468517	15.5360684	14.6209871	13.0352078	36
37	20.5705254	19.1425788	17.8622398	16.7112873	15.6739985	14.7367803	13.1170166	37
38	20.8410874	19.3678642	18.0499902	16.8678927	15.8047379	14.8460192	13.1934735	38
39	21.1024999	19.5844848	18.2296557	17.0170407	15.9286615	14.9490747	13.2649285	39
40	21.3550723	19.7927739	18.4015844	17.1590864	16.0461247	15.0462969	13.3317089	40
41	21.5991037	19.9930518	18.5661095	17.2943680	16.1574642	15.1380159	13.3941204	41
42	21.8348828	20.1856267	18.7235498	17.4232076	16.2629992	15.2245433	13.4524490	42
43	22.0626887	20.3707949	18.8742103	17.5459120	16.3630324	15.3061729	13.5069617	43
44	22.2827910	20.5488413	19.0183831	17.6627733	16.4578506	15.3831820	13.5579081	44
45	22.4954503	20.7200397	19.1563474	17.7740698	16.5477257	15.4558321	13.6055216	45
46	22.7009181	20.8846536	19.2883707	17.8800665	16.6329154	15.5243699	13.6500202	46
47	22.8994378	21.0429361	19.4147088	17.9810157	16.7136639	15.5890282	13.6916077	47
48	23.0912443	21.1951309	19.5356065	18.0771578	16.7902027	15.6500266	13.7304744	48
49	23.2765645	21.3414720	19.6512981	18.1687217	16.8627514	15.7075723	13.7667986	49
50	23.4556179	21.4821846	19.7620078	18.2559255	16.9315179	15.7618606	13.8007463	50
51	23.6286163	21.6174852	19.8679500	18.3389766	16.9966994	15.8130761	13.8324732	51
52	23.7957645	21.7475819	19.9693302	18.4180730	17.0584829	15.8613925	13.8621245	52
53	23.9572604	21.8726749	20.0663447	18.4934028	17.1170454	15.9069741	13.8898359	53
54	24.1132951	21.9929567	20.1591815	18.5651456	17.1725549	15.9499755	13.9157345	54
55	24.2640532	22.1086122	20.2480206	18.6334720	17.2251705	15.9905430	13.9399388	55
56	24.4097133	22.2198194	20.3330340	18.6985447	17.2750431	16.0288141	13.9625596	56
57	24.5504476	22.3267494	20.4143866	18.7605188	17.3223158	16.0649190	13.9837006	57
58	24.6864228	22.4295668	20.4922360	18.8195417	17.3671239	16.0989802	14.0034585	58
59	24.8177998	22.5284296	20.5667330	18.8757540	17.4095961	16.1311134	14.0219238	59
60	24.9447341	22.6234900	20.6380220	18.9292895	17.4498542	16.1614277	14.0391812	60

TABLE 18.—*The annuity for n years which 1 will buy or the annuity needed to discharge a debt of 1 in n years with interest.*

$$\frac{1}{a_{\overline{n}|}}=\frac{i}{1-v^n}.$$

Years.	1½%.	1¾%.	2%.	2¼%.	2½%.	2¾%.	3%.	Years.
1	1.0150000	1.0175000	1.0200000	1.0225000	1.0250000	1.0275000	1.0300000	1
2	0.5112779	0.5131630	0.5150495	0.5169376	0.5188272	0.5207183	0.5226108	2
3	0.3433830	0.3450675	0.3467547	0.3484446	0.3501372	0.3518324	0.3535304	3
4	0.2594448	0.2610324	0.2626238	0.2642189	0.2658179	0.2674206	0.2690271	4
5	0.2090893	0.2106214	0.2121584	0.2137002	0.2152469	0.2167983	0.2183546	5
6	0.1755252	0.1770226	0.1785258	0.1800350	0.1815500	0.1830708	0.1845975	6
7	0.1515562	0.1530306	0.1545120	0.1560003	0.1574954	0.1589975	0.1605064	7
8	0.1335840	0.1350429	0.1365098	0.1379846	0.1394674	0.1409580	0.1424564	8
9	0.1196098	0.1210581	0.1225154	0.1239817	0.1254569	0.1269410	0.1284339	9
10	0.1084342	0.1098754	0.1113265	0.1127877	0.1142588	0.1157397	0.1172305	10
11	0.0992938	0.1007304	0.1021779	0.1036365	0.1051060	0.1065863	0.1080775	11
12	0.0916800	0.0931138	0.0945596	0.0960174	0.0974871	0.0989687	0.1004621	12
13	0.0852404	0.0866728	0.0881184	0.0895769	0.0910483	0.0925325	0.0940295	13
14	0.0797233	0.0811556	0.0826020	0.0840623	0.0855365	0.0870246	0.0885263	14
15	0.0749444	0.0763774	0.0778255	0.0792885	0.0807665	0.0822592	0.0837666	15
16	0.0707651	0.0721996	0.0736501	0.0751166	0.0765990	0.0780971	0.0796109	16
17	0.0670797	0.0685162	0.0699698	0.0714404	0.0729278	0.0744319	0.0759525	17
18	0.0638058	0.0652449	0.0667021	0.0681772	0.0696701	0.0711806	0.0727087	18
19	0.0608785	0.0623206	0.0637818	0.0652618	0.0667606	0.0682780	0.0698139	19
20	0.0582457	0.0596912	0.0611567	0.0626421	0.0641471	0.0656717	0.0672157	20
21	0.0558655	0.0573146	0.0587848	0.0602757	0.0617873	0.0633194	0.0648718	21
22	0.0537033	0.0551564	0.0566314	0.0581282	0.0596466	0.0611864	0.0627474	22
23	0.0517308	0.0531880	0.0546681	0.0561710	0.0576964	0.0592441	0.0608139	23
24	0.0499241	0.0513857	0.0528711	0.0543802	0.0559128	0.0574686	0.0590474	24
25	0.0482635	0.0497295	0.0512204	0.0527360	0.0542759	0.0558400	0.0574279	25
26	0.0467320	0.0482027	0.0496992	0.0512213	0.0527688	0.0543412	0.0559383	26
27	0.0453153	0.0467908	0.0482931	0.0498219	0.0513769	0.0529578	0.0545642	27
28	0.0440011	0.0454815	0.0469897	0.0485253	0.0500879	0.0516774	0.0532932	28
29	0.0427788	0.0442642	0.0457784	0.0473208	0.0488913	0.0504894	0.0521147	29
30	0.0416392	0.0431298	0.0446499	0.0461993	0.0477776	0.0493844	0.0510193	30
31	0.0405743	0.0420701	0.0435963	0.0451528	0.0467390	0.0483545	0.0499989	31
32	0.0395771	0.0410781	0.0426106	0.0441742	0.0457683	0.0473926	0.0490466	32
33	0.0386414	0.0401478	0.0416865	0.0432572	0.0448594	0.0464925	0.0481561	33
34	0.0377619	0.0392736	0.0408187	0.0423966	0.0440068	0.0456488	0.0473220	34
35	0.0369336	0.0384508	0.0400022	0.0415873	0.0432056	0.0448565	0.0465393	35
36	0.0361524	0.0376751	0.0392329	0.0408252	0.0424516	0.0441113	0.0458038	36
37	0.0354144	0.0369426	0.0385068	0.0401064	0.0417409	0.0434095	0.0451116	37
38	0.0347161	0.0362499	0.0378206	0.0394275	0.0410701	0.0427476	0.0444593	38
39	0.0340546	0.0355940	0.0371711	0.0387854	0.0404362	0.0421226	0.0438439	39
40	0.0334271	0.0349721	0.0365558	0.0381774	0.0398362	0.0415315	0.0432624	40
41	0.0328311	0.0343817	0.0359719	0.0376009	0.0392679	0.0409720	0.0427124	41
42	0.0322643	0.0338206	0.0354173	0.0370536	0.0387288	0.0404418	0.0421917	42
43	0.0317247	0.0332867	0.0348899	0.0365336	0.0382169	0.0399387	0.0416981	43
44	0.0312104	0.0327781	0.0343879	0.0360390	0.0377304	0.0394610	0.0412299	44
45	0.0307198	0.0322932	0.0339096	0.0355681	0.0372675	0.0390069	0.0407852	45
46	0.0302512	0.0318304	0.0334534	0.0351192	0.0368268	0.0385749	0.0403625	46
47	0.0298034	0.0313884	0.0330179	0.0346911	0.0364067	0.0381636	0.0399605	47
48	0.0293750	0.0309657	0.0326018	0.0342823	0.0360060	0.0377716	0.0395778	48
49	0.0289648	0.0305612	0.0322040	0.0338918	0.0356235	0.0373977	0.0392131	49
50	0.0285717	0.0301739	0.0318232	0.0335184	0.0352581	0.0370409	0.0388655	50
51	0.0281947	0.0298027	0.0314586	0.0331610	0.0349087	0.0367001	0.0385338	51
52	0.0278329	0.0294466	0.0311091	0.0328188	0.0345745	0.0363744	0.0382172	52
53	0.0274854	0.0291049	0.0307739	0.0324909	0.0342545	0.0360630	0.0379147	53
54	0.0271514	0.0287767	0.0304523	0.0321765	0.0339480	0.0357649	0.0376256	54
55	0.0268302	0.0284613	0.0301434	0.0318749	0.0336542	0.0354795	0.0373491	55
56	0.0265211	0.0281580	0.0298466	0.0315853	0.0333724	0.0352061	0.0370845	56
57	0.0262234	0.0278661	0.0295612	0.0313071	0.0331020	0.0349440	0.0368311	57
58	0.0259366	0.0275850	0.0292867	0.0310398	0.0328424	0.0346927	0.0365885	58
59	0.0256601	0.0273143	0.0290224	0.0307827	0.0325931	0.0344515	0.0363559	59
60	0.0253934	0.0270534	0.0287680	0.0305353	0.0323534	0.0342200	0.0361330	60

TABLE 18.—*The annuity for n years which 1 will buy or the annuity needed to discharge a debt of 1 in n years with interest*—Continued.

$$\frac{1}{a_{\overline{n|}}}=\frac{i}{1-v^n}.$$

Years.	3½%.	4%.	4½%.	5%.	5½%.	6%.	7%.	Years.
1	1.0350000	1.0400000	1.0450000	1.0500000	1.0550000	1.0600000	1.0700000	1
2	0.5264005	0.5301961	0.5339976	0.5378049	0.5416180	0.5454369	0.5530918	2
3	0.3569342	0.3603485	0.3637734	0.3672086	0.3706541	0.3741098	0.3810517	3
4	0.2722511	0.2754901	0.2787437	0.2820118	0.2852945	0.2885915	0.2952281	4
5	0.2214814	0.2246271	0.2277916	0.2309748	0.2341764	0.2373964	0.2438907	5
6	0.1876682	0.1907619	0.1938784	0.1970175	0.2001790	0.2033626	0.2097958	6
7	0.1635445	0.1666096	0.1697015	0.1728198	0.1759644	0.1791350	0.1855532	7
8	0.1454767	0.1485278	0.1516097	0.1547218	0.1578640	0.1610359	0.1674678	8
9	0.1314460	0.1344930	0.1375745	0.1406901	0.1438395	0.1470222	0.1534865	9
10	0.1202414	0.1232909	0.1263788	0.1295046	0.1326678	0.1358680	0.1423775	10
11	0.1110920	0.1141490	0.1172482	0.1203889	0.1235707	0.1267929	0.1333569	11
12	0.1034840	0.1065522	0.1096662	0.1128254	0.1160292	0.1192770	0.1259020	12
13	0.0970616	0.1001437	0.1032754	0.1064558	0.1096843	0.1129601	0.1196509	13
14	0.0915707	0.0946690	0.0978203	0.1010240	0.1042791	0.1075849	0.1143449	14
15	0.0868251	0.0899411	0.0931138	0.0963423	0.0996256	0.1029628	0.1097946	15
16	0.0826848	0.0858200	0.0890154	0.0922699	0.0955825	0.0989521	0.1058577	16
17	0.0790431	0.0821985	0.0854176	0.0886999	0.0920420	0.0954448	0.1024252	17
18	0.0758168	0.0789933	0.0822369	0.0855462	0.0889199	0.0923565	0.0994126	18
19	0.0729403	0.0761386	0.0794073	0.0827450	0.0861501	0.0896209	0.0967530	19
20	0.0703611	0.0735818	0.0768761	0.0802426	0.0836793	0.0871846	0.0943929	20
21	0.0680366	0.0712801	0.0746006	0.0779961	0.0814648	0.0850046	0.0922890	21
22	0.0659321	0.0691988	0.0725457	0.0759705	0.0794712	0.0830456	0.0904058	22
23	0.0640188	0.0673091	0.0706825	0.0741368	0.0776696	0.0812785	0.0887139	23
24	0.0622728	0.0655868	0.0689870	0.0724709	0.0760358	0.0796790	0.0871890	24
25	0.0606740	0.0640120	0.0674390	0.0709525	0.0745494	0.0782267	0.0858105	25
26	0.0592054	0.0625674	0.0660214	0.0695643	0.0731931	0.0769044	0.0845610	26
27	0.0578524	0.0612385	0.0647195	0.0682919	0.0719523	0.0756972	0.0834257	27
28	0.0566027	0.0600130	0.0635208	0.0671225	0.0708144	0.0745926	0.0823919	28
29	0.0554454	0.0588799	0.0624146	0.0660455	0.0697686	0.0735796	0.0814487	29
30	0.0543713	0.0578301	0.0613915	0.0650514	0.0688054	0.0726489	0.0805864	30
31	0.0533724	0.0568554	0.0604435	0.0641321	0.0679167	0.0717922	0.0797969	31
32	0.0524415	0.0559486	0.0595632	0.0632804	0.0670952	0.0710023	0.0790729	32
33	0.0515724	0.0551036	0.0587445	0.0624900	0.0663347	0.0702729	0.0784081	33
34	0.0507597	0.0543148	0.0579819	0.0617554	0.0656296	0.0695984	0.0777967	34
35	0.0499984	0.0535773	0.0572705	0.0610717	0.0649749	0.0689739	0.0772340	35
36	0.0492842	0.0528869	0.0566058	0.0604345	0.0643664	0.0683948	0.0767153	36
37	0.0486133	0.0522396	0.0559840	0.0598398	0.0637999	0.0678574	0.0762369	37
38	0.0479821	0.0516319	0.0554017	0.0592842	0.0632722	0.0673581	0.0757951	38
39	0.0473878	0.0510608	0.0548557	0.0587646	0.0627799	0.0668938	0.0753868	39
40	0.0468273	0.0505235	0.0543432	0.0582782	0.0623203	0.0664615	0.0750091	40
41	0.0462982	0.0500174	0.0538616	0.0578223	0.0618909	0.0660589	0.0746596	41
42	0.0457983	0.0495402	0.0534087	0.0573947	0.0614893	0.0656834	0.0743359	42
43	0.0453254	0.0490899	0.0529824	0.0569933	0.0611134	0.0653331	0.0740359	43
44	0.0448777	0.0486645	0.0525807	0.0566163	0.0607613	0.0650061	0.0737577	44
45	0.0444534	0.0482625	0.0522020	0.0562617	0.0604313	0.0647005	0.0734996	45
46	0.0440511	0.0478821	0.0518447	0.0559282	0.0601218	0.0644149	0.0732600	46
47	0.0436692	0.0475219	0.0515073	0.0556142	0.0598313	0.0641477	0.0730374	47
48	0.0433065	0.0471807	0.0511886	0.0553184	0.0595585	0.0638977	0.0728307	48
49	0.0429617	0.0468571	0.0508872	0.0550397	0.0593023	0.0636636	0.0726385	49
50	0.0426337	0.0465502	0.0506022	0.0547767	0.0590615	0.0634443	0.0724599	50
51	0.0423216	0.0462589	0.0503323	0.0545287	0.0588350	0.0632388	0.0722937	51
52	0.0420243	0.0459821	0.0500768	0.0542945	0.0586219	0.0630462	0.0721390	52
53	0.0417410	0.0457192	0.0498347	0.0540733	0.0584213	0.0628655	0.0719951	53
54	0.0414709	0.0454691	0.0496052	0.0538644	0.0582325	0.0626960	0.0718611	54
55	0.0412132	0.0452312	0.0493875	0.0536669	0.0580546	0.0625370	0.0717363	55
56	0.0409673	0.0450049	0.0491811	0.0534801	0.0578870	0.0623877	0.0716201	56
57	0.0407325	0.0447893	0.0489851	0.0533034	0.0577290	0.0622474	0.0715118	57
58	0.0405081	0.0445840	0.0487990	0.0531363	0.0575801	0.0621157	0.0714109	58
59	0.0402937	0.0443884	0.0486222	0.0529780	0.0574396	0.0619920	0.0713169	59
60	0.0400886	0.0442019	0.0484543	0.0528282	0.0573071	0.0618757	0.0712292	60

TABLE 19.—*Bid on a bond for $100 to realize a given net income, interest payable semiannually.*

INTEREST 3½%.

Net income.	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.
3.00	102.31	104.29	106.00	107.48	108.75	109.85
3.10	101.84	103.42	104.77	105.93	106.92	107.78
3.20	101.38	102.55	103.55	104.41	105.14	105.76
3.30	100.91	101.69	102.35	102.91	103.39	103.79
3.40	100.46	100.84	101.17	101.44	101.68	101.87
3.50	100.00	100.00	100.00	100.00	100.00	100.00
3.60	99.55	99.17	98.85	98.58	98.36	98.17
3.70	99.09	98.34	97.71	97.19	96.76	96.39
3.80	98.65	97.52	96.59	95.82	95.19	94.66
3.90	98.20	96.71	95.49	94.48	93.65	92.96
4.00	97.75	95.91	94.40	93.16	92.14	91.31
4.10	97.31	95.12	93.33	91.86	90.67	89.70
4.20	96.87	94.33	92.27	90.59	89.23	88.12
4.30	96.44	93.55	91.22	89.34	87.82	86.59
4.40	96.00	92.78	90.19	88.11	86.44	85.09
4.50	95.57	92.02	89.18	86.90	85.08	83.63
4.60	95.14	91.26	88.18	85.72	83.76	82.20
4.70	94.71	90.51	87.19	84.55	82.46	80.80
4.80	94.28	89.77	86.21	83.40	81.19	79.44
4.90	93.86	89.04	85.25	82.28	79.95	78.12
5.00	93.44	88.31	84.30	81.17	78.73	76.82

INTEREST 4%.

Net income.	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.
3.00	104.61	108.58	112.01	114.96	117.50	119.69
3.10	104.14	107.69	110.73	113.34	115.58	117.50
3.20	103.67	106.80	109.47	111.75	113.70	115.35
3.30	103.20	105.92	108.23	110.19	111.85	113.27
3.40	102.74	105.05	107.00	108.66	110.05	111.23
3.50	102.28	104.19	105.80	107.15	108.29	109.24
3.60	101.82	103.33	104.60	105.67	106.56	107.30
3.70	101.36	102.49	103.43	104.21	104.87	105.41
3.80	100.90	101.65	102.27	102.78	103.21	103.56
3.90	100.45	100.82	101.13	101.38	101.59	101.76
4.00	100.00	100.00	100.00	100.00	100.00	100.00
4.10	99.55	99.19	98.89	98.64	98.45	98.28
4.20	99.11	98.38	97.79	97.31	96.92	96.61
4.30	98.66	97.58	96.71	96.00	95.43	94.97
4.40	98.22	96.79	95.64	94.72	93.97	93.37
4.50	97.78	96.01	94.59	93.45	92.54	91.81
4.60	97.35	95.23	93.55	92.21	91.14	90.29
4.70	96.91	94.47	92.53	90.99	89.77	88.80
4.80	96.48	93.71	91.52	89.79	88.42	87.35
4.90	96.05	92.95	90.52	88.61	87.11	85.93
5.00	95.62	92.21	89.53	87.45	85.82	84.55

INTEREST 4½%.

Net income.	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.
3.00	106.92	112.88	118.01	122.44	126.25	129.54
3.10	106.44	111.96	116.69	120.75	124.23	127.22
3.20	105.96	111.05	115.39	119.09	122.26	124.95
3.30	105.49	110.15	114.11	117.47	120.32	122.74
3.40	105.02	109.26	112.84	115.87	118.43	120.59
3.50	104.55	108.38	111.59	114.30	116.57	118.48
3.60	104.08	107.50	110.36	112.75	114.75	116.43
3.70	103.62	106.64	109.15	111.24	112.98	114.42
3.80	103.16	105.78	107.95	109.74	111.23	112.47
3.90	102.70	104.93	106.77	108.28	109.53	110.56
4.00	102.25	104.09	105.60	106.84	107.86	108.69
4.10	101.79	103.25	104.45	105.42	106.22	106.87
4.20	101.34	102.43	103.31	104.03	104.62	105.09
4.30	100.89	101.61	102.19	102.66	103.05	103.35
4.40	100.44	100.80	101.09	101.32	101.51	101.66
4.50	100.00	100.00	100.00	100.00	100.00	100.00
4.60	99.56	99.21	98.93	98.70	98.52	98.38
4.70	99.12	98.42	97.86	97.43	97.08	96.80
4.80	98.68	97.64	96.82	96.17	95.66	95.26
4.90	98.25	96.87	95.79	94.94	94.27	93.75
5.00	97.81	96.10	94.77	93.72	92.91	92.27

TABLE 19.—*Bid on a bond for $100 to realize a given net income, interest payable semiannually*—Continued.

INTEREST 5%.

Net income.	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.
3.00	109.22	117.17	124.02	129.92	135.00	139.38
3.10	108.74	116.23	122.65	128.16	132.89	136.93
3.20	108.26	115.30	121.31	126.44	130.81	134.55
3.30	107.78	114.38	119.99	124.75	128.79	132.22
3.40	107.30	113.47	118.68	123.08	126.80	129.94
3.50	106.83	112.56	117.39	121.45	124.86	127.72
3.60	106.35	111.67	116.12	119.84	122.95	125.55
3.70	105.88	110.78	114.86	118.26	121.08	123.44
3.80	105.42	109.91	113.62	116.70	119.26	121.37
3.90	104.95	109.04	112.40	115.18	117.47	119.35
4.00	104.49	108.18	111.20	113.68	115.71	117.38
4.10	104.03	107.32	110.01	112.20	113.99	115.45
4.20	103.57	106.48	108.84	110.75	112.31	113.57
4.30	103.12	105.64	107.68	109.33	110.66	111.74
4.40	102.67	104.81	106.54	107.93	109.04	109.94
4.50	102.22	103.99	105.41	106.55	107.46	108.19
4.60	101.77	103.18	104.30	105.19	105.91	106.47
4.70	101.32	102.37	103.20	103.86	104.38	104.80
4.80	100.88	101.57	102.12	102.55	102.89	103.16
4.90	100.44	100.78	101.05	101.27	101.43	101.56
5.00	100.00	100.00	100.00	100.00	100.00	100.00

INTEREST 6%.

Net income.	5 years.	10 years.	15 years.	20 years.	25 years.	30 years.
3.50	111.38	120.94	128.98	135.74	141.43	146.20
3.60	110.89	120.01	127.63	134.01	139.34	143.81
3.70	110.41	119.08	126.30	132.30	137.30	141.47
3.80	109.93	118.16	124.98	130.63	135.30	139.18
3.90	109.46	117.25	123.68	128.98	133.34	136.94
4.00	108.98	116.35	122.40	127.36	131.42	134.76
4.10	108.51	115.46	121.13	125.76	129.54	132.63
4.20	108.04	114.58	119.88	124.19	127.70	130.54
4.30	107.58	113.70	118.65	122.65	125.89	128.50
4.40	107.11	112.83	117.43	121.14	124.11	126.51
4.50	106.65	111.97	116.23	119.65	122.38	124.56
4.60	106.19	111.12	115.05	118.18	120.67	122.66
4.70	105.73	110.28	113.88	116.74	119.00	120.80
4.80	105.28	109.44	112.73	115.32	117.36	118.98
4.90	104.83	108.61	111.59	113.92	115.76	117.20
5.00	104.38	107.79	110.47	112.55	114.18	115.45
5.25	103.26	105.78	107.72	109.22	110.38	111.27
5.50	102.16	103.81	105.06	106.02	106.75	107.31
5.75	101.07	101.88	102.49	102.95	103.29	103.55
6.00	100.00	100.00	100.00	100.00	100.00	100.00

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